LINDELÖF PERFECT FUNCTIONS

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Abstract. In this paper we introduce a new notion of the perfect functions in the topological spaces, which called lindelöf perfect functions.

Also we study the images and inverse images of certain topological properties under these functions. We derive some related results. Finally some product theorems obtained concerning these concept .

1. Introduction and Preliminaries

Firstly , dear reader, we present to you a basic definitions and a brief introductory summary of the perfect functions in the single topological spaces and some studies about these conjugations in the topological spaces and the important results that have been reached according to these studies.

A continuous function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is said to be perfect if X is a Hausdorff space, f is closed and the fibers $f^{-1}(y)$ are compact subsets of X.

In1952, Vainstein [12] for the first time introduced the class of perfect functions in the realm of metric spaces.

Independently, perfect functions were introduced and studied (in the realm of locally compact spaces) by Leray in 1950 and Baurkbaki in1951.

Later several mathematicians worked on perfect functions and proved several results concerning it is effect on different topological spaces.

For instance Hdeib(1982) [9].

In this paper we introduce the notions and concepts of new perfect functions in the topological spaces, it is called Lindelöf perfect functions.

Let us recall known definitions which will be used in the sequel.

Definition 1.1. [6]Let (X, τ) be a topological space, and let $U \subseteq \tau$, be a collection of open subsets of X. We say U is an open cover of X if $X = \cup\, U$.

If U is an open cover of X and $V \subseteq U$ is a subcollection of U that is also an open cover of X, we say V is a subcover of U.

Definition 1.2. [6]A topological space (X, τ) is said to be compact if every open cover of

X has a finite subcover.

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Definition 1.3. [7]A topological space (X, τ) is said to be Lindelöf if every open cover of X has a countable subcover.

Definition 1.4. [5]A topological space (X, τ) is Hausdorff if for any $x, y \in X$ with $x \neq y$, there exist open sets U containing x and Vcontaining y such that $U \cap V = \phi$.

Definition 1.5. [6]A topological space (X, τ) is said to be locally compact if every point has an open nbhd with compact closure.

Definition 1.6. [7]Let (X, T_X) and (Y, T_Y) be topological spaces. A function $f : X \to Y$ is said to be continuous , if the inverse image of every open subset of Y is open in X.

In other words, if $V \in T_Y$, then its inverse image $f^{-1}(V) \in T_X$.

Definition 1.7. [5]Let (X, T_X) and (Y, T_Y) be topological spaces. A function $f : X \to Y$ is said to be open, if for any open set U in X, the image $f(U)$ is open in Y.

Definition 1.8. [6]Let (X, T_X) and (Y, T_Y) be topological spaces. A function $f : X \to Y$ is said to be closed, if for any closed set F in X, the image $f(F)$ is open in Y.

Definition 1.9. [5]A topological space (X, τ) is said to be countably compact if every countable open cover of X has a finite subcover.

Definition 1.10. [6]A topological space (X, τ) is said to be a paracompact space if every open cover of X has an open refinement that is locally finite.

Theorem 1.11. *[7]Let $X = (X, \tau)$ be a Hausdörff space, then every Lindelöf subset is closed .*

Theorem 1.12. *[5]A closed proper subset of a Lindelöf space is Lindelöf .*

2. Lindelöf Perfect Functions

In this section, we will introduce the concept of Lindelö perfect functions in topological spaces, and introduce some of their properties, and relate it to other spaces.

Definition 2.1. A function $f : (X, \tau) \to (Y, \sigma)$ is called Lindelöf perfect, if f is continuous, closed, and for each $y \in Y$, $f^{-1}(y)$ is Lindelöf .

Corollary 2.2. *In above definition if $f^{-1}(y)$ is countable then f is Lindelö perfect function.*

Example 2.3. Let $f : (R, \tau_{ind}) \longrightarrow (R, \tau_{ind})$ be the identity function ,where τ_{ind} is indiscrete topology, then f is Lindelö perfect function.

Since f is continuous , closed and for each $y \in Y$ any open cover $\tilde{U}$ of $f^{-1}(y)$ must be contains X because the only non empty open set in $(R\ , \tau_{ind})$ is X. Hence $\{X\}$ is a countable subcover of $\tilde{U}$. Hence $f^{-1}(y)$ is Lindelöf .

Theorem 2.4. *if the function* $f : (X, \tau) \to (Y, \sigma)$*is perfect functions , then f is Lindelö perfect functions, but the converse need not be true.*

Proof. It is obvious, that f is continuous, closed, and for each $y \in Y$, $f^{-1}(y)$ is compact, then $f^{-1}(y)$ is Lindelöf. Hence f is Lindelö perfect functions. □

Corollary 2.5. *if the function* $f : (X, \tau) \to (Y, \sigma)$*is Lindelö perfect functions , then f need not be perfect functions.*

Example 2.6. let $f : (R, \tau_s) \to (R, \tau_s)$ is Lindelö perfect functions, since in $R_{Sorgenfrey}$, let $U = \{[-x; x) : x > 0\}$ is a cover that has many countable subcovers, for example V=$\{(-n, n) : n \in N\}$ is a subcover of U, so for each $y \in R_{Sorgenfrey}$, $f^{-1}(y)$ is countable, thus $f^{-1}(y)$ is Lindelöf. Note however that$V$=$\{(-n, n) : n \in N\}$ no finite subcollection of U can cover R, so $f^{-1}(y)$ is not compact. Hence f is not perfect functions.

Theorem 2.7. *if the function* $f : (X, \tau) \to (Y, \sigma)$ *is Lindelö perfect functions , and* $f^{-1}(y)$ *is countably compact, then f is perfect functions.*

Proof. It is clearly, that f is continuous, closed, and for each $y \in Y$, $f^{-1}(y)$ is Lindelöf, and $f^{-1}(y)$ is countably compact, therefor $f^{-1}(y)$ is compact, since every Lindelöf and countably compact is compact. Hence f is perfect functions. □

Theorem 2.8. *If* $f : (X, \tau) \longrightarrow (Y, \sigma)$ *is a Lindelö perfect function, then for every Lindelöf subset* $Z \subseteq Y$*, the inverse image* $f^{-1}(Z)$ *is a Lindelöf .*

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of (X, τ) , since f is a Lindelö perfect function, then $\forall y \in Y$, $f^{-1}(y)$ is Lindelöf subset of X. So there exists a countable subsets Λ_y of Λ ,s.t $f^{-1}(y) \subseteq \left(\bigcup_{\alpha \in \Lambda_y} \{V_\alpha : \alpha \in \Lambda_y\} \right)$ where $\{V_\alpha : \alpha \in \Lambda_y\}$ is an open subsets of X . Now , let $O_y = Y - f\ (X - \bigcup_{\alpha \in \Lambda_y} V_\alpha)$ is a an open subset of Y containing y ,then $y \in O_y$ .Since $f^{-1}(O_y) \subseteq \bigcup_{\alpha \in \Lambda_y} V_\alpha$ then $\tilde{O} = \{O_y : y \in\ Y\ \}$ is an open cover of Y. Hence , $\tilde{O}$ is open cover of Z . Since Z is Lindelof ,$\tilde{O}$ has a countable subcover$\left(\bigcup_{i=1} (O_{y_i}) \right)$ and $Z \subseteq \left(\bigcup_{i=1} (O_{y_i}) \right)$ .Thus, $f^{-1}\ (Z) \subseteq \bigcup_{i=1} f^{-1}(O_{y_i})$ subset of a union of countable subset of $\tilde{U}$, i.e $f^{-1}\ (Z)$ is Lindelöf . □

Corollary 2.9. *A Lindelöf space is inverse invariant under Lindelöf perfect functions.*

Corollary 2.10. *The composition of two Lindelöf perfect functions is a Lindelö perfect function.*

Proposition 2.11. *If the composition* $g \circ f$ *of a continuous functions,* $f : (X, \tau) \overset{onto}{\to} (Y, \sigma)$ *and* $g : (Y, \sigma) \overset{onto}{\to} (Z, \rho)$ *is a closed then the function* $g : (Y, \sigma) \overset{onto}{\to} (Z, \rho)$ *is a closed.*

Proof. Let A be a closed subset of Y,then $f^{-1}(A)$ is a closed subset of X. Since $g \circ f$ is a closed, then $g(f\ \ (f^{-1}(A))) = g(A)$ is aclosed subset of Z .Thus g is a closed. □

Theorem 2.12. *If the composition function $g \circ f$ of a continuous funcions $f : (X, \tau) \overset{onto}{\to} (Y, \sigma)$, $g : (Y, \sigma) \overset{onto}{\to} (Z, \rho)$ is a Lindelö perfect, then the function $g : (Y, \sigma) \overset{onto}{\to} (Z, \rho)$ is a Lindelö perfect.*

Proof. For every $z \in Z$, $g^{-1}(z) = f\ ((g \circ f\ )^{-1}(z))$ is a Lindelöf subset of Y, because $g \circ f$ is a Lindelö perfect. □

Since g is a closed by proposition 2.11, we get that g is Lindelo perfect .

Theorem 2.13. *If $f : (X, \tau) \overset{onto}{\to} (Y, \sigma)$ is a closed function ,then for any $B \subset Y$ the restriction $f_B : f^{-1}(B) \to B$ is a closed .*

Proof. Let $B \subset Y$ .Consider the function $f : (X, \tau_1) \to (Y, \sigma_1)$. let A be a a closed subset of X. Then $f_B\ (A \bigcap f^{-1}(B)) = f(A) \bigcap B$ is aclosed subset of B. □

Thus $f_B : f^{-1}(B) \to B$ is a closed.

Theorem 2.14. *If $f : (X, \tau) \overset{onto}{\to} (Y, \sigma)$ is a Lindelö perfect function ,then for any $B \subset Y$ the restriction $f_B : f^{-1}(B) \to B$ is a Lindelö perfect.*

Proof. The proof follows directly from theorem 2.13. □

Theorem 2.15. *If $f : (X, \tau) \overset{onto}{\longrightarrow} (Y, \sigma)$ is a Lindelö perfect function ,where (X, τ) is a Lindelöf, and (Y, σ) is a Hausdörff ,then f is a closed .*

Proof. If A is a closed subset of (X, τ) ,then it is a Lindelöf because (X, τ) is a Lindelöf. Since f is a continuous , $f(A)$ is a Lindelöf subset of (Y, σ). □

Since (Y, σ) is Hausdörff, then $f(A)$ is a closed subset of (Y, σ) . Hence the result.

Theorem 2.16. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a continuous function from a Hausdörff space (X, τ) in to a locally compact space (Y, σ).*

Then the following are equivalent :

(i) f is a Lindelö perfect function,

(ii) For every Lindelöf subset $Z \subset Y$ the set $f^{-1}(Z)$ is a Lindelöf subset of X .

Proof. $(i) \Rightarrow (ii)$: the proof follows from theorem 2.13. □

$(ii) \Rightarrow (i)$: It is suffices to show that f is a closed function , i.e the function $f : (X, \tau) \to (Y, \sigma)$ is closed function. Let A be a closed subset of X, and y be a cluster point of $f(A)$.Suppose $y \notin f(A)$.Since Y is locally compact, there is a open set V containing y s.t $CL\,(V)$ is compact, and so Lindelöf.

Now, $f^{-1}(CL\,(V) \bigcap f(A)) = f^{-1}(CL\,(V)) \bigcap A$ .By using (ii) $f^{-1}(CL\,(V))$ is Lindelöf and A is a closed , Lindelöf subset .

Also , $f(f^{-1}(CL\,(V)) \bigcap A\ ) = CL\,(V) \bigcap f(A)$ is a Lindelöf subset which is closed.

Now,$V - CL\,(V) \bigcap\ f(A)\ = U$ is an open set containing p and $U\ \bigcap f(A) = \phi$,which contradicts the fact that p is a cluster point. Hence $p \in f(A)$,i.e $f(A)$ is a closed.

Thus $f : (X, \tau) \to (Y, \sigma)$ is a closed function.

Theorem 2.17. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a continuous bijection function. If* (Y, σ) *is a Hausdörff space, and* (X, τ) *is a Lindelöf, then* f *is a homeomorphic function.*

Proof. It's enough to show that f is a closed.Let F be a closed proper subset of X, and hence F is proper Lindelöf, by using theorem [2.14] □

Hence , $f(F)$ is a Lindelof, but (Y, σ) is a Hausdörff space, by theorem [2.15], $f(F)$ is a closed, i.e. f is a homeomorphic function.

Definition 2.18. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called a strongly function,

if for every open cover $\tilde{U} = \{U_\alpha : \alpha \in \Lambda\}$ of X there exists an open cover

$\tilde{V} = \{V_\gamma : \gamma \in \Gamma\}$of Y, s.t $f^{-1}(V\ ) \subseteq \bigcup \{U_\alpha : \alpha \in \Lambda\ :\ \Lambda$ is a countable subset of $\Lambda\} \,\forall\, V \in \tilde{V}$.

Theorem 2.19. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a strongly onto function, then* (X, τ) *is a*

Lindelöf, if (Y, σ) *is so.*

Proof. Let $\tilde{U}\ = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover (X, τ).Since f is a strongly function, there exists an open cover $\tilde{V} = \{V_\gamma : \gamma \in \Gamma\}$ of (Y, σ), □

such that $f^{-1}(V\ ) \subseteq \bigcup \{U_\alpha : \alpha \in \Lambda\ :\ \Lambda$ is a countable subset of $\Lambda\} \,\forall\, V \in \tilde{V}$.

but (Y, σ) is a Lindelöf , so there exists a countable subset Γ s.t $Y = \bigcup\limits_{\gamma \in\ \Gamma_1} V_\gamma$. Hence $X = \bigcup\limits_{\gamma \in\ \Gamma} f^{-1}(V_\gamma)$.So each $f^{-1}(V_\gamma)$ contains in a countable number of members of $\tilde{U}$ .Thus X is a Lindelöf.

Theorem 2.20. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a Lindelö perfect function such that* $\forall y \in Y$, $f^{-1}(y)$ *is a countably compact,. If* (Y, σ) *is a countably compact , then* (X, τ) *is so.*

Proof. Let $\tilde{U} = \{U_\alpha{:}\alpha \in \Lambda\}$ be an open cover of (X, τ). Since f is a Lindelö perfect function, then $\forall y \in Y$, $f^{-1}(y)$ is a Lindelöf, $\exists$ a countable subsets Λ_y of Λ , □

s.t $f^{-1}(y) \subseteq \bigcup\limits_{\alpha \in \Lambda_y} \{V_\alpha : \alpha \in \Lambda_y\}$ where $\{V_\alpha : \alpha \in \Lambda_y\}$ is an open subsets of X . Now , $Oy(\alpha, y) = Y - f(X - \bigcup\limits_{\alpha \in \Lambda_y} V_\alpha)$ is an open set containing y .

Also , $f^{-1}(O_y(\alpha, y) \subseteq \bigcup_{\alpha\in\Lambda_y} V_\alpha$ .Let $\tilde{O} = \{O_y(\alpha, y) \ : \ y \in \ Y\}$,then $\tilde{O}$ a countable cover of Y . Since (Y, σ) is countably compact , $\tilde{O}$ has a countable subcover say , $\tilde{O}^* = \{O_y(\alpha_i, y) \ : \ i \in N \ , \ y \in \ Y\}$. So ($X, \tau) = \bigcup_{i\in N} f^{-1}(O_y(\alpha_i, y))$.Hence (X, τ) is a countably compact.

The following theorem shows that a paracompactness is an inverse invariant under Lindelö perfect function.

Theorem 2.21. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a Lindelö perfect function . If* (Y, σ) *is a regular paracompact space then* (X, τ) *is so.*

Proof. Let $\tilde{U} = \{U_\alpha{:}\alpha \in \Lambda\}$ be an open cover of (X, τ) , since f is a Lindelö perfect function, then $\forall y \in Y, \ f^{-1}(y)$ is a Lindelöf, $\exists$ a countable subsets Λ_y of Λ , □

s.t $f^{-1}(y) \subseteq \bigcup_{\alpha\in\Lambda_y} \{V_\alpha : \alpha \in \Lambda_y\}$, where $\{V_\alpha : \alpha \in \Lambda_y\}$ is an open. Let$O_y = Y - f \ (X - \bigcup_{\alpha\in\Lambda_y} V_\alpha)$ is an open set containing y, where $f^{-1}(O_y) \subseteq \bigcup_{\alpha\in\Lambda_y} V_\alpha$.

Now , $\tilde{O} = \{O_y : y \in \ Y \ \}$ is an open cover of Y .Since (Y, σ) is a paracompact $\tilde{O}$ has an open locally finite parallel refinement say $\tilde{H} = \{H_B : B \in \Gamma \ \}$ where $\{H_B : B \in \Gamma \ \}$ is a locally finite parallel refinement of $\{O_y : y \in \ Y \ \}$.

Let $S = \{f^{-1}(H_B) \bigcap V_{\alpha_i} \ : B \in \Gamma, \alpha_i \in \Lambda_y\}$ then S is an open locally finite parallel refinement of $\{V_\alpha : \alpha \in \Lambda\}$, then S is an open locally finite parallel refinement of $\tilde{U}$, so (X, τ) is a paracompact regular space.

Theorem 2.22. *The Hausdroff space is invariant under Lindelö perfect functions.*

Proof. Let (X, τ) be a Hausdroff space, $f : (X, \tau) \to (Y, \sigma)$ be a Lindelö perfect function, and $y_1 \neq y_2$ in (Y, σ),then $f^{-1}(y_1)$, $f^{-1}(y_2)$ are disjoint and Lindelöfness subset of (X, τ) . Since (X, τ) be a Hausdroff space ,there exists a neighborhoods U , V in X , and s.t $f^{-1}(y_1) \subseteq U$, $f^{-1}(y_2) \subseteq V$ and $U \bigcap V = \phi$.Now , the sets $Y - f(X - U)$ is an □

open subset in (Y, σ) containing y_1 and $Y - f(X - V)$ is an open subset in (Y, σ) containing y_2 , s.t $[Y - f(X - U) \bigcap Y - f(X - V)] = Y - [f(X - U) \bigcup f(X - V)]$

$= Y - f(X - U \bigcap V \ ) = Y - f(X) = \phi$.Hence (Y, σ) is a Hausdroff space.

Theorem 2.23. *Let* $(X, \tau), (Y, \sigma)$ *be any two bitopological spaces .If* (X, τ) *is a Lindelöf , and* (Y, σ) *is compact ,then the projection function,* $\pi : (X \times Y \ , \tau \times \sigma) \to (Y, \sigma)$ *is closed.*

Proof. since (X, τ) is a Lindelöf and (Y, σ) is compact ,then $(X \times Y \ , \tau \times \sigma)$ is Lindelöf, so the projection functions:$\pi : (X \times Y \ , \tau \times \sigma) \to (Y, \sigma)$ closed functions. □

Corollary 2.24. *Let* $(X, \tau), (Y, \sigma)$ *be any two bitopological spaces .If* (X, τ) *is a Lindelöf and* (Y, σ) *is Lindelöf , then the projection function,* $\pi : (X \times Y \ , \tau \times \sigma) \to (Y, \sigma)$ *need not be closed.*

Proof. since (X, τ) is a Lindelöf and (Y, σ) is Lindelöf ,then $(X \times Y, \tau \times \sigma)$ need not be Lindelöf. For example (R, τ_s) (the sorgenfrey topology) is Lindelöf, and $(R^2, \tau_s \times \tau_s)$ (the sorgenfrey plane) is not Lindelöf, so $\pi : (R^2, \tau_s \times \tau_s) \to (Y, \sigma)$ is not closed. $\square$

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