A New Study on Tri-Lindelöf Spaces

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Abstract: - This paper defines new covering properties in tri-topological spaces called tri-Lindelöf space and the properties of this topological property and its relationship with some other types of tri-topological spaces will be studied. The effect of some types of functions on tri-Lindelöf spaces will be studied. This paper also investigates the necessary conditions through which the tri-topological space is reduced into a single topological space. Many and varied illustrative examples will be discussed and many well-known facts and theorems are generalized concerning Lindelöf spaces.

Key-Words: - Topological space, Tri-topological space, Tri-Lindelöf space, S- compact space, T- compact space, C- compact space, S- Lindelöf space, T- Lindelöf space, C- Lindelöf space

1 Introduction

The study of Tri-topological spaces is considered a generalization of the same study in bi-topological spaces, which in turn was a generalization of single topological spaces. This study is based on choosing any non-empty set W with three topologies $\tau_W^1, \tau_W^2, \tau_W^3$ defined on W which is called the tri-topological space denoted by $(W, \tau_W^1, \tau_W^2, \tau_W^3)$. To clarify what we started in our introduction, we present the following: The first person to discuss the idea of bi-topological spaces was, [1], who built this idea on a non-empty set W defined by two topologies τ_W^1 and τ_W^2, and this was denoted by $(W, \tau_W^1, \tau_W^2, \tau_W^3)$.He then proceeded to generalize the definition of a number of topological spaces to bi-topological spaces, including Hausdörff, regular and normal spaces. This idea gained the attention of researchers of that era, and their efforts led to the definition of pairwise compact spaces with a comprehensive study of their features by scientists, [2], ‘ Cooke and Reilly,[3], and Bersan,[4]. In the 1980s, specifically in 1983, a study was conducted that focused on Lindelöff’s theories in bi-topological spaces, during which the two scientists, Fora and Hdeib,[5], reached distinctive results and generalizations that led to the development and confirmation of many topological concepts in this field, while supporting those concepts with illustrative examples that illustrate the relationship between those concepts. See also, [6] and [7]. This is a simple summary of the concepts, research, and studies in bi-topological spaces that formed a research base for researchers later to generalize them in tri-topological spaces and elsewhere. It will also be presented in our introduction to the topic of the study. The study and discussion of the basic principles of the tri-topological spaces began in the year 2000 by Martin Kover,[8], where many important separation axioms were defined in tri-topological spaces, and those concepts were clarified by setting various and distinctive examples. These studies encourage many researchers to conduct numerous studies in that field. In 2011 new types of separation axioms were defined in tri-topological spaces based on the open sets in those spaces called $123-T_0, 123-T_1$ and $123-T_2$ spaces, see Hameed and Abid ,[9]. These concepts paved the way for their definitions of some types of separation axioms in tri-spaces that do not depend on open sets, in those spaces, but rather on other types of open sets in tri-topological spaces, where three new types of separation axioms define b−open sets and called them $123b-T_0, 123b-T_1$ and $123b-T_2$ spaces, to benefit more see, [9]. Also, In 2011, the doctoral dissertation submitted by the researcher Palaniammal,[10], includes in its main topic a detailed study of the tri-topological spaces and discussed them. Many results were reached, the most important of which is the development of definitions of the tri$-\alpha-$continuous functions and tri$-\beta-$continuous functions. Also In the same year 2011, Sweedy and Hassan,[11], introduced the concept of $\delta-$continuous functions with a detailed study of their properties in tri−topological spaces. Subsequent to the above and in 2016, Tapi. et. al.,[12], a new study was conducted aimed at defining two types of open sets in tri−topological spaces: semi-open sets and pre-open sets among their basic properties. Two important concepts of continuous functions with many associated properties were also discussed namely: tri−semi−continuous and tri−pri−continuous functions. In 2017, Priyadharshini and Parvathi,[13], studied the main topological characteristics of tri$-b-$open sets and tri−b−closed sets and used them to define tri$-b-$continuous functions. They studied the properties of these functions and their nature and connected them to other types of functions of tri−topological spaces. Distinguished scientific efforts In 2017, we summarize as follows: a new type of open sets has been discussed in tri−topological spaces by the researchers, Tapi and Sharma,[14], dubbed it $\alpha T-$open sets where properties that they studied. It was used in defining one of the types of continuous functions associated with it called $\alpha T-$functions and obtained some of their properties. The efforts of the researchers continued in the field of research in the tri−topological spaces so in 2017, a scientific paper was presented entitled "Soft tri−topological spaces" through the researcher Hassan,[15], in which good results were reached and generalizations were issued. For more studies on this subject note the following researcher's references [16] and [17]. In 2018, Sharma. et. al., [18], give the concept of fuzzy connectedness as a new study in tri−topological spaces. They also discussed several separation properties in this topic. In 2019, Hassan,[19], presented an important study that discussed the main basics of the subject of (fuzzy soft tri−topological space) and drew a number of conclusions and recommendations. In 2021, Arasi. et. al., [20], a new topic has been addressed in tri-topological spaces based on tri$-g-$closed sets and $\hat{g}$-closed set which is tri$-\hat{g}-$continuous functions. Also, during this research, the following topics were discussed which are: quasi-tri-$\hat{g}$, perfectly tri$-\hat{g}$ and tri$-\hat{g}-$continuous functions. It is worth noting that qualitative conclusions were reached in this work. The topic of neutrosophic Tri−topological space was studied in 2021 by Das and Pramanik,[21]. This study aimed to generalize the topic of neutrosophic topological spaces and several results and characteristics related to this topic. Further, different types of open and closed sets were defined in the neutrosophic tri−spaces and their different properties were compared and discussed. Also, during the year 2021, an important study was conducted entitled "neutrosophic soft tri−topological spaces" that highlighted the discussion of a variety of topological characteristics in "neutrosophic soft topological spaces" and its generalization in tri−topological spaces. The following sets have been defined and their basic characteristics studied through this study: neutrosophic soft tri−open and tri-closed sets, see Dados and Demiraip,[22]. In 2022, Annalakshmi. et al.,[23], studied the concepts of supra tri−topological spaces and gave the definitions of some open sets such as t−open sets and supra tri−semi−pre open sets in supra tri−topological spaces. In 2023, fuzzy soft tri−pre−open sets were investigated and studied by Hassan and Farhan,[24]. This study focused mostly on studying the basic properties of those sets. In this paper, First, a new definition of open covers in tri-topological spaces and their types was presented, where these types were used to define several types of tri−topological spaces such as S−compact spaces, T−compact spaces, C−compact spaces, S− Lindelöf spaces, T−Lindelöf spaces, C− Lindelöf spaces, S−countably compact spaces, T− countably compact spaces and C−countably compact spaces.

second, The relationship between these concepts has been studied and many illustrative examples of these concepts have been developed. these examples also show the relationship between these concepts.

Third, many theories were proved and introduced to these new concepts, the most important of which was the study of the effect of some types of functions on them, and a number of results and characteristics were reached on this subject.

Fourth and finally, the necessary and sufficient conditions that are required to reduce tri-topological spaces into a single topological space have been studied.

In this paper, we use the following letters: the letters τ −closure, τ −interior of a set B will be denoted by $CL(B)$, $Int(B)$ respectively. The product of τ_W^1 and τ_W^2 will be denoted by $\tau_W^1 \times \tau_W^2$.

Let ($\mathbb{R}, \mathbb{Z}, \mathbb{N}, \mathbb{Q}$) denote the set of all (real, integer, natural, and rational) numbers, respectively. Let $(\tau_{dis}, \tau_{ind}, \tau_u, \tau_s, \tau_{coc}, \tau_{cof}, \tau_l, \tau_r)$ denote the (discrete, indiscrete, usual, Sorgenfrey line, cocountable, cofinite, left-ray, and right-ray) topologies, respectively.

2 Tri−Lindlöf spaces

In this section, we will discuss the concept of tri−Lindlöf topological spaces and extract some of their characteristics and their relationship to several other types of tri−topological spaces with illustrative examples for each case. Also, many theories on this subject will be discussed and proven.

Definition 2.1. A cover $\mathcal{U}$ of a tri−topological space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is called $\tau_W^1\tau_W^2\tau_W^3$ −open if $\mathcal{U} \subset \tau_W^1 \cup \tau_W^2 \cup \tau_W^3$. If in addition, $\mathcal{U}$ contains at least one non-empty member of each $\tau_W^i : i = 1,2,3$, then $\mathcal{U}$ is called τ_W^{123} −open cover.

Definition 2.2. (1) A tri−topological space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is called S−compact if every $\tau_W^1\tau_W^2\tau_W^3$ −open the cover of the space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ has a finite subcover.

(2) A tri−topological space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is called T−$c$ompact if every τ_W^{123} −open the cover of the space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ has a finite subcover.

(3) A tri−topological space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is said to be C− C-compact if every $\tau_W^i - an$ open cover of W, $i = 1,2,3$ has a τ_W^j −finite subcover, $j = 1,2,3$ and τ_W^k −finite subcover $k = 1,2,3$, where $i \neq j \neq k$.

Example 2.1 The tri-topological space $(\mathbb{R}, \tau_{dis}, \tau_{ind}, \tau_u)$ is T−compact, since every τ^{123} −open cover $\mathcal{U}$ for $\mathbb{R}$ must contain $\mathbb{R}$. So $\mathcal{U}' = \{\mathbb{R}\}$ is a finite subcover of $\mathcal{U}$ for $\mathbb{R}$. But $(\mathbb{R}, \tau_{dis}, \tau_{ind}, \tau_u)$ is not S−compact since $\mathcal{U} = \{\{x\}: x \in \mathbb{R}\}$ is a $\tau_{dis} - an$ open cover of $\mathbb{R}$, so it is a $\tau^1\tau^2\tau^3 -$ open cover which has no finite subcover.

Definition 2.3 (**1**) A tri−topological space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is called S−Lindelöf if every $\tau_W^1\tau_W^2\tau_W^3$ −open the cover of the space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ has a countable subcover.

(2) A tri−topological space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is called T−Lindelöf if every τ_W^{123} − open the cover of the space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ has a countable subcover.

(3) A tri−topological space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is said to be C−Lindelöf if every $\tau_W^i - an$ open cover of $W, i = 1,2,3$ has a τ_W^jcountable subcover, $j = 1,2,3$ and τ_W^k countable subcover $k = 1,2,3$, where $i \neq j \neq k$.

Example 2.2 The tri−topological space $(\mathbb{R}, \tau_{dis}, \tau_{coc}, \tau_{cof})$ is T− Lindelöf.

Example 2.3 The tri−topological space $(\mathbb{N}, \tau_{dis}, \tau_{ind}, \tau_{cof})$ is S−Lindelöf and T−Lindelöf space.

Example 2.4 $(\mathbb{N}, \tau_{dis}, \tau_{coc}, \tau_{cof})$ is not C−Lindelöf.

Example 2.5 $(\mathbb{N}, \tau_{dis}, \tau_{dis}, \tau_{dis}) \equiv (\mathbb{N}, \tau_{dis})$ is C−Lindelöf.

Theorem 2.1 A tri−topological space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is S−Lindelöf iff (W, τ) is Lindelöf space, where $\tau = \tau^1 \vee \tau^2 \vee \tau^3$ is the least upper bound topology of τ^1,$\tau^2$and τ^3.

Proof: Assume that (W, τ) is Lindelöf space, let $\mathcal{U}$ be a $\tau^1\tau^2\tau^3$ −open cover of $(W, \tau_W^1, \tau_W^2, \tau_W^3)$, then

$\mathcal{U} \subseteq \tau^1 \cup \tau^2 \cup \tau^3 \subseteq \tau^1 \vee \tau^2 \vee \tau^3$. But $(W, \tau^1 \vee \tau^2 \vee \tau^3)$ is lindelöf space, so $\mathcal{U}$ has a countable subcover.

Remark 2.1 It is clear that if $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is C− Lindelöf then each of $(W, \tau_W^1), (W, \tau_W^2)$ and (W, τ_W^3) is Lindelöf.

Remark 2.2 A tri−topological space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ has a particular topological property if τ^1, τ^2 and τ^3 have this property. For instance, $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is said to be T_1 −space if all $(W, \tau_W^1), (W, \tau_W^2)$ and (W, τ_W^3) are T_1 −spaces.

Example 2.6 The tri−topological space $(R, \tau_{dis}, \tau_{cof}, \tau_u)$ is T_1 −space.

Theorem 2.2 A tri−topological space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is S−lindelöf iff it is lindelöf and T−lindelöf.

Proof: Assume that $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is S− lindelöf. Since any T−open or τ_W^1 −open or τ_W^2 −open or $\tau_W^3 - an$ open cover of $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is a $\tau_W^1 \tau_W^2 \tau_W^3 -$ open cover, then the result is as follows.

Conversely, if a $\tau_W^1 \tau_W^2 \tau_W^3$ −open cover of $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is not T-open, then it is τ_W^1 −open or τ_W^2 −open or τ_W^3 −open cover of $(W, \tau_W^1, \tau_W^2, \tau_W^3)$. Since $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is Lindelöf, then the result is as follows.

Example 2.7 The tri−topological space $(\mathbb{R}, \tau_s, \tau_u, \tau_{cof})$ is $S-$Lindelöf. So it is $T-$ Lindelöf and Lindelöf. But not $C-$ Lindelöf, since the τ_{u-}open cover $\{(-n, n): n \in \mathbb{N}\}$ of $\mathbb{R}$ has no τ_{cof} −open countable subcover.

Example 2.8 Let $W = \mathbb{R}, \mathcal{B}_1 = \{W, \{w\}: w \in W - \{1\}\}, \mathcal{B}_2 = \{W, \{w\}: w \in W - \{2\}\}, \mathcal{B}_3 = \{W, \{w\}: w \in W - \{3\}\}$, let τ^1, τ^2 and τ^3 be the topologies defined on W, which are generated by the bases $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}_3$, respectively. Then $(W, \tau^1, \tau^2, \tau^3)$ is C−Lindelöf. Since any τ^1 −open cover of W or any $\tau^2 - an$ open cover of W or any $\tau^3 - an$ open cover of W must contain W, so $\{W\}$ is a countable subcover of any $\tau^i - an$ open cover of W, $i = 1,2,3$. It is clear that $(W, \tau^1, \tau^2, \tau^3)$ is not $T-$ Lindelöf, for the τ^{123} −open cover $\{\{w\}: w \in W\}$ has no countable subcover, also, $(W, \tau^1, \tau^2, \tau^3)$ is Lindelöf, but not S−Lindelöf since it is not $T-$ Lindelöf.

Example 2.9 The tri−topological space $(\mathbb{R}, \tau_{dis}, \tau_{coc}, \tau_l)$ is T−Lindelöf but not S−Lindelöf.

Example 2.10 The tri−topological space $(\mathbb{R}, \tau_u, \tau_{ind}, \tau_{cof})$ is T−Lindelöf but not S− Lindelöf. because $(\mathbb{R}, \tau_u)$ is not Lindelöf.

Example 2.11 Consider the three typologies τ^1, τ^2 and τ^3 on $\mathbb{R}$, generated by the basis $\mathcal{B}_1 = \{(-\infty, b): b > 0\} \cup \{\{a\}: a > 0\}, \mathcal{B}_2 = \{(b, \infty): b < 0\} \cup \{\{a\}: a < 0\}$ and $\mathcal{B}_3 = \{\{x\}: x \in \mathbb{R}\}$, respectively. Then $(\mathbb{R}, \tau^1, \tau^2, \tau^3)$ is T− Lindelöf but not Lindelöf. It is clear that $(\mathbb{R}, \tau^1, \tau^2, \tau^3)$ is not S− lindelöf, since the τ^3 −open cover $\{\{x\}: x \in \mathbb{R}\}$ of $\mathbb{R}$ which exactly a $\tau^1\tau^2\tau^3 -$ open cover but it has no countable subcover.

Definition 2.4 (1) A tri−topological space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is called T−countably compact if every countably $\tau_W^{123} -$ open cover of the space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ has a finite subcover.
(2) A tri−topological space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is called S−countably compact if every countably $\tau_W^1 \tau_W^2 \tau_W^3$ −open cover of the space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ has a finite subcover.
(3) A tri-topological space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is called C−countably compact if every countably $\tau_W^i -$ open cover of the space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$, $i = 1,2,3$, has a finite τ_W^j −open subcover $j = 1,2,3$, and a finite τ_W^k −open subcover $k = 1,2,3, i \neq j \neq k$.

Example 2.12 The tri−topological space $(\mathbb{R}, \tau_{dis}, \tau_{coc}, \tau_{ind})$ is T−Lindelöf space which is neither T−countably compact nor T−compact.

Example 2.13 Let $W = \mathbb{R}$, $\mathcal{B}_1 = \{\{w\}: w \in \mathbb{R} - \{0\}\}, \mathcal{B}_2 = \{\{w\}: w \in \mathbb{R} - \{1\}\}, \mathcal{B}_3 = \{\{w\}: w \in \mathbb{R} - \{2\}\}$ and let τ^1, τ^2, and τ^3 be the topologies on $\mathbb{R}$ which are generated by $\mathcal{B}_1, \mathcal{B}_2$, and $\mathcal{B}_3$ respectively, then $(\mathbb{R}, \tau^1, \tau^2, \tau^3)$ is C− Lindelöf; since for any $\tau^1 -$ open cover of $\mathbb{R}$ or $\tau^2 -$ open cover of $\mathbb{R}$ or $\tau^3 -$ open cover of $\mathbb{R}$ must contains $\mathbb{R}$ as a member. So $\{\mathbb{R}\}$ is a countable subcover of each τ^i −open cover, $i = 1,2,3$. However, $(\mathbb{R}, \tau^1, \tau^2, \tau^3)$ is not T−Lindelöf, for the $\tau^{123} -$ open cover $\{\{w\}: w \in \mathbb{R}\}$ of $\mathbb{R}$ has no countable subcover. It is clear that $(\mathbb{R}, \tau^1, \tau^2, \tau^3)$ is neither T−compact nor T−countably compact. Also, $(\mathbb{R}, \tau^1, \tau^2, \tau^3)$ is C−compact and C-countably compact. We can see also, $(\mathbb{R}, \tau^1, \tau^2, \tau^3)$ it is not S-Lindelöf so not S-compact.

It is easy to prove the following theorem

Theorem 2.3 **(1)** Every T−(respectively S, C) compact space is T−(respectively S, C) countably compact and T−(respectively S, C) Lindelöf space.
(2) Every T−(respectively S, C) countably compact T−(respectively S, C) Lindelöf is T−(respectively S, C) compact.

Proof **(1)** Assume that $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is a T−compact space. Let $\mathcal{U}$ be a $\tau_W^{123} - an$ open cover of W then $\mathcal{U}$ has a finite subcover $\mathcal{U}'$, so $\mathcal{U}'$ is a countable subcover for $\mathcal{U}$.
The remaining parts in **(1)** are intelligibly evident in a similar fashion.
(2) Assume that $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is a T−countably compact, T−Lindelöf space. Let $\mathcal{U}$ be a τ_W^{123} −open cover of W. Since $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is T−Lindelöf, $\mathcal{U}$ has a countable subcover $\mathcal{U}'$. Since $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is a T−countably compact then $\mathcal{U}'$ has a finite subcover for $\mathcal{U}''$.
The remaining parts in **(2)** are intelligibly evident in a similar fashion.

Theorem 2.4 If a tri−topological space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is a hereditary Lindelöf space, then it is S−Lindelöf.

Proof Let $\mathcal{U} = \{u_\alpha : \alpha \in \Delta_1\} \cup \{v_\beta : \beta \in \Delta_2\} \cup \{w_\gamma : \gamma \in \Delta_3\}$ be a $\tau_W^1 \tau_W^2 \tau_W^3 -$ an open cover of a non-empty set W, where $u_\alpha \in \tau_W^1, \forall \alpha \in \Delta_1, v_\beta \in \tau_W^2, \forall \beta \in \Delta_2$ and $w_\gamma \in \tau_W^3, \forall \gamma \in \Delta_3$. Since $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is a hereditary Lindelöf, then $\mathcal{U}_1 = \cup\{u_\alpha : \alpha \in \Delta_1\}$ is $\tau_W^1 -$ Lindelöf, then there exists a countable set $\Delta_1^* \subseteq \Delta_1$ such that $\mathcal{U}_1 = \cup \{u_\alpha : \alpha \in \Delta_1^*\}$. Similarly, $\mathcal{U}_2 = \cup \{v_\beta : \beta \in \Delta_2^*\}$ is $\tau_W^2 -$ Lindelöf, then there exists a countable set $\Delta_2^* \subseteq \Delta_2$ such that $\mathcal{U}_2 = \cup \{v_\beta : \beta \in \Delta_2^*\}$. Also, $\mathcal{U}_3 = \cup\{w_\gamma : \gamma \in \Delta_3\}$ is $\tau_W^3 -$ Lindelöf, then there exists a countable set $\Delta_3^* \subseteq \Delta_3$ such that $\mathcal{U}_3 = \cup \{v_\gamma : \gamma \in \Delta_3^*\}$. Now, $\{u_\alpha : \alpha \in \Delta_1^*\} \cup \{v_\beta : \beta \in \Delta_2^*\} \cup \{w_\gamma : \gamma \in \Delta_3^*\}$ is a countable subcover of $\mathcal{U}$ for W. Hence $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is S- Lindelöf.

Corollary 2.1 Every second countable tri-topological space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is S- Lindelöf.

Proof Since every second countable space is hereditary Lindelöf. Hence the result.

Theorem 2.5 A τ_W^i −closed proper subset of a S−Lindelöf space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is τ_W^j −Lindelöf and τ_W^k −Lindelöf, where $(i \neq j \neq k, i, j, k = 1,2,3)$.

Proof Let $\mathcal{F}$ be a nonempty τ_W^i −closed proper subset of W and let $\mathcal{U} = \{u_\alpha : \alpha \in \Delta\}$ be τ_W^j −open cover t $\{w_1, w_2, w_3, \ldots\}$ of $W - \mathcal{F}$ such that $\{u_\alpha : \alpha \in \Delta^*\} \cup \{v(w_1), v(w_2), v(w_3), \ldots\}$ is a countable subcover of $\mathcal{U}$for W. Since $v(w_i) \subset W - \mathcal{F}$, then $v(w_i) \cap \mathcal{F} = \phi$ for all i then $(\bigcup_{i=1}^{\infty} v(w_i)) \cap \mathcal{F} = \phi$ and then $\{u_\alpha : \alpha \in \Delta^*\}$ is a countable subcover for $\mathcal{F}$, hence $\mathcal{F}$ is a τ_W^j −Lindelöf. Similarly, we can prove that every τ_W^i −closed proper subset of W is τ_W^j −Lindelöf.of $\mathcal{F}$. Since $W - \mathcal{F}\ is\ \tau_W^i$ −open, then for each $w \in W - \mathcal{F}$ there exists a τ_W^i −open set $v(w)$ such that $w \in v(w) \subseteq W - \mathcal{F}$. Since $\mathcal{F} \neq \phi$ and $\mathcal{F} \neq W$, then $\mathcal{U} = \{u_\alpha : \alpha \in \Delta\} \cup \{v(w) : w \in W - \mathcal{F}\}$ is a $\tau_W^{ij} -$ an open cover of W hence it is a $\tau_W^1 \tau_W^2 \tau_W^3 -$ an open cover of an S−Lindelöf space W. Thus there exists a countable set Δ^* of Δ and a countable set

Corollary 2.2 A τ_W^i −closed proper subset of an S−S-compact space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is τ_W^j −compact and τ_W^k −compact, where $(i \neq j \neq k, i, j, k = 1,2,3)$.

Definition 2.7 A family $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ of nonvoid subsets of a tri−topological space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is said to be $\tau_W^1 \tau_W^2 \tau_W^3$ −closed if every member F of $\mathcal{F}$ is τ_W^1 −closed or τ_W^2 −closed or τ_W^3 −closed for all $\alpha \in \Delta$. If $\mathcal{F}$ contains members C_1, C_2, C_3 such that C_1 is τ_W^1 −closed and C_2 is τ_W^2 −closed and C_3 is τ_W^3 −closed, then it is called T−closed.

Theorem 2.6 A tri topological space $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ is T −Lindelöf if and only if every T −closed family with countable intersection property has a nonempty intersection.

Proof Let $\mathcal{A} = \{F_\alpha : \alpha \in \Delta\} \cup \{C_\beta : \beta \in \Gamma\} \cup \{K_\xi : \xi \in \Upsilon\}$ be a family of T−closed proper subset of W with countable intersection property. Suppose that $\cap \mathcal{A} = \phi$. Now, $\mathcal{U} = \{W - F_\alpha : \alpha \in \Delta\} \cup \{W - C_\beta : \beta \in \Gamma\} \cup \{W - K_\xi : \xi \in \Upsilon\}$ is a $\tau_W^{123} - an$ open cover of W because $\{\cup (W - F_\alpha) : \alpha \in \Delta\} \cup \{\cup (W - C_\beta) : \beta \in \Gamma\} \cup \{\cup (W - K_\xi) : \xi \in \Upsilon\} = \{W - \cap F_\alpha : \alpha \in \Delta\} \cup \{W - \cap C_\beta : \beta \in \Gamma\} \cup \{W - \cap K_\xi : \xi \in \Upsilon\} = W - \{(\cap F_\alpha : \alpha \in \Delta) \cap (\cap C_\beta : \beta \in \Gamma) \cap (\cap K_\xi : \xi \in \Upsilon)\} = W - \cap \mathcal{A} = W$. But W is T−Lindelöf so $\mathcal{U}$has a countable subcover $\mathcal{U}' =$

$\{W-F_{\alpha_1}, W-F_{\alpha_2}, W-F_{\alpha_3}, \dots\} \cup \{W-C_{\beta_1}, W-C_{\beta_2}, W-C_{\beta_3}, \dots\} \cup \{W-K_{\xi_1}, W-K_{\xi_2}, W-K_{\xi_3}, \dots\}$. Hence $\bigcup\mathcal{U}=W$ i.e $W-\{(\bigcap F_{\alpha_i}: i\in\mathbb{N}) \cap (\bigcap C_{\beta_i}: i\in\mathbb{N}) \cap (\bigcap K_{\xi_i}: i\in\mathbb{N})\} = W. So\ \{(\bigcap F_{\alpha_i}: i\in\mathbb{N}) \cap (\bigcap C_{\beta_i}: i\in\mathbb{N}) \cap (\bigcap K_{\xi_i}: i\in\mathbb{N})\} = \phi$, which is a contradiction. Hence $\bigcap\mathcal{A}=\phi$.
Conversely; assume that W is not T−Lindelöf. Let $\mathcal{U}=\{F_\alpha:\alpha\in\Delta\} \cup \{C_\beta:\beta\in\Gamma\} \cup \{K_\xi:\xi\in\Upsilon\}$ be a $\tau_W^{123}-$ an open cover of W which has no countable subcover. Now $\mathcal{A}=\{W-F_\alpha:\alpha\in\Delta\} \cup \{W-C_\beta:\beta\in\Gamma\} \cup \{W-K_\xi : \xi\in\Upsilon\}$ is a T−closed family which has the countable intersection property; by assumption $\bigcap\mathcal{A}\neq\phi$ i.e. $W-\{\bigcup F_\alpha:\alpha\in\Delta\} \cup \{\bigcup C_\beta:\beta\in\Gamma\} \cup \{\bigcup K_\xi:\xi\in\Upsilon\}\neq\phi$. Which is a contradiction since $\bigcup\mathcal{U}=W$.

Theorem 2.7 A tri-topological space $(W,\tau_W^1,\tau_W^2,\tau_W^3)$ is S−Lindelöf space if and only if every $\tau_W^1\tau_W^2\tau_W^3-a$ closed family with countable intersection property has a nonempty intersection.

Proof The proof is similar to the last theorem.

Theorem 2.8 Let $(W,\tau_W^1,\tau_W^2,\tau_W^3)$ be a C−compact and $(Z,\tau_z^1,\tau_z^2,\tau_z^3)$ be a C−Lindelöf. Then $(W\times Z,\tau_w^1\times\tau_z^1,\tau_w^2\times\tau_z^2,\tau_w^3\times\tau_z^3)$ is C−Lindelöf.

Definition 2.6 A tri−topological space $(W,\tau_W^1,\tau_W^2,\tau_W^3)$ is said to be T−Hausdörff if for any two distinct points w and z in W there exists a τ_W^i −open set u_w of w and τ_W^j −open set u_z of z such that $u_w\cap u_z=\phi$, where $(i,j=1,2,3, i\neq j)$.

Remark 2.3 If a tri−topological space $(W,\tau_W^1,\tau_W^2,\tau_W^3)$ is T−Hausdörff, then τ_W^1,τ_W^2 and τ_W^3 are T_1 −spaces.

Theorem 2.9 If a tri−topological space $(W,\tau_W^1,\tau_W^2,\tau_W^3)$ is T−Hausdörff, then for each $w\in W$ we have $\{w\}=\bigcap\{cl_i u_\alpha: u_\alpha$ is a τ_W^j −neighbourhood of $w, i,j=1,2,3, i\neq j\}$

Proof Assume that $(W,\tau_W^1,\tau_W^2,\tau_W^3)$ is T−Hausdörff space. Let $w,z\in W$ such that $w\neq z$ and $z\in \bigcap\{cl_i u_\alpha: u_\alpha\ is\ a\ \tau_W^j-$ neighbourhood of $w, i,j=1,2,3, i\neq j\}$. Since W is T−Hausdörff, then there is a τ_W^i −open set u_w of w and a τ_W^j −open set u_z of z such that $u_w\cap u_z=\phi$. Hence $w\in u_z\subseteq X-u_w$, then $cl_i u_z\subseteq X-u_w$, but $z\notin X-u_w$, then $z\notin cl_i u_z$ so $z\notin\bigcap\{cl_i u_\alpha: u_\alpha\ is\ a\ \tau_W^j-$ neighbourhood of $w, i,j=1,2,3, i\neq j\}$ which is a contradiction. Hence $\{w\}=\bigcap\{cl_i u_\alpha: u_\alpha\ is\ a\ \tau_W^j-neighbourhood\ of\ w, i,j=1,2,3, i\neq j\}$

Recall that: A topological space $(W,\tau_W^1,\tau_W^2,\tau_W^3)$ is said to be P-space if the intersection of any countable number of open sets is open.

Theorem 2.10 If $(W,\tau_W^1,\tau_W^2,\tau_W^3)$ is a T−Housdörf and P−space, then any τ_W^i −Lindelöf proper subset of X is τ_W^j −closed $(i,j=1,2,3, i\neq j)$.

Proof Let B be a τ_W^i −Lindelöf proper subset of W and $w\in W-B$, then by theorem [2.9] we have $\{w\}=\bigcap\{cl_i u_\alpha: u_\alpha$ is a τ_W^j- neighbourhood of $w, i,j=1,2,3, i\neq j\}$. Since $B\subseteq X-\{w\}$, then $B\subseteq W-\bigcap\{cl_i u_\alpha: u_\alpha\ is\ a\ \tau_W^j-$ neighbourhood of $w, i,j=1,2,3, i\neq j\} = \bigcup_{\alpha\in\Delta}\{W-cl_i u_\alpha\}$, therefore, $\{W-cl_i u_\alpha:\alpha\in\Delta\}$ is a τ_W^i-an open cover of B. so there exists a countable subset Δ_1 subset of Δ such that $\{W-cl_i u_\alpha:\alpha\in\Delta_1\}$ is a τ_W^i-an open cover of B. Let $U=\bigcap_{\alpha\in\Delta_1}u_\alpha$, then u is a τ_W^j −open set such that $w\in U$ and $U\subseteq W-B$. Hence $w\in U\subseteq W-B$, this implies that B is a τ_W^j −closed $(i,j=1,2,3, i\neq j)$.

Corollary 2.3 If $(W,\tau_W^1,\tau_W^2,\tau_W^3)$ is T−Hausdörff , then every τ_W^i-a compact subset of W is τ_W^j −closed $(i,j=1,2,3, i\neq j)$.

Proof Using the same technique as the above theorem.

Definition 2.7 A function $f:(W,\tau_W^1,\tau_W^2,\tau_W^3)\to(Z,\tau_Z^1,\tau_Z^2,\tau_Z^3)$ is said to be T−continuous (T−open, T−closed, T−homomorphism, respectively) if $f:(W,\tau_W^1)\to(Z,\tau_Z^1)$, $f:(W,\tau_W^2)\to(Z,\tau_Z^2)$ and $f:(W,\tau_W^3)\to(Z,\tau_Z^3)$are continuous (open, closed, homomorphism, respectively).

Theorem 2.11 Let $f:(W,\tau_W^1,\tau_W^2,\tau_W^3)\to(Z,\tau_Z^1,\tau_Z^2,\tau_Z^3)$ be a T−continuous onto function, then

(1) If $(W,\tau_W^1,\tau_W^2,\tau_W^3)$ is T−Lindelöf (S−Lindelöf, C−Lindelöf ,respectively), then $(Z,\tau_Z^1,\tau_Z^2,\tau_Z^3)$ is T−Lindelöf (S−Lindelöf, C−Lindelöf ,respectively).

(2) If $(W,\tau_W^1,\tau_W^2,\tau_W^3)$ is T-compact(S−compact, C−compact,respectively), then $(Z,\tau_Z^1,\tau_Z^2,\tau_Z^3)$ is T-compact(S−compact, C−compact,respectively).

Proof

(1) Let $\mathcal{U}=\{u_\alpha:\alpha\in\Delta\}\cup\{v_\alpha:\alpha\in\Delta\}\cup\{w_\alpha:\alpha\in\Delta\}$ be a $\tau_Z^{123}-an$ open cover of Z where $u_\alpha\in\tau_Z^1, v_\alpha\in\tau_Z^2$ and $w_\alpha\in\tau_Z^3,\alpha\in\Delta$. Since f is T−continuous and onto, then $\{f^{-1}(u_\alpha):\alpha\in\Delta\}\cup\{f^{-1}(v_\alpha\ ):\alpha\in\Delta\}\cup\{f^{-1}(w_\alpha):\alpha\in\Delta\}\, is\ a\ \tau_W^{123}$ −open cover of W. Since $(W,\tau_W^1,\tau_W^2,\tau_W^3)$ is T−Lindelöf, then there exists a countable subset Δ_1 of Δ such that $\{f^{-1}(u_\alpha):\alpha\in\Delta_1\}\cup\{f^{-1}(v_\alpha\ ):\alpha\in\Delta_1\}\cup\{f^{-1}(w_\alpha):\alpha\in\Delta_1\}$ is a countable sub-cover of $\mathcal{U}$ for W, So $\mathcal{U}^*=\{u_\alpha:\alpha\in\Delta_1\}\cup\{v_\alpha:\alpha\in\Delta_1\}\cup\{w_\alpha:\alpha\in\Delta_1\}$ is a countable subcover of $\mathcal{U}$ for Z. Hence Z is T−Lindelöf. the remaining part of **(1)** Is easy and similarly proved.

(2) The proof is similar to that in the statement **(1)**.

Theorem 2.12 Let $f:(W,\tau_W^1,\tau_W^2,\tau_W^3)\to(Z,\tau_Z^1,\tau_Z^2,\tau_Z^3)$ a be T-continuous onto the map, then

(1) If f is one-to-one, $(Z,\tau_Z^1,\tau_Z^2,\tau_Z^3)$ is T−Housdörf, P−space and $(W,\tau_W^1,\tau_W^2,\tau_W^3)$ is S−Lindelöf, then f is T−homomorphism.

(2) If f is one-to-one, $(Z,\tau_Z^1,\tau_Z^2,\tau_Z^3)$ is T−Housdörf and $(W,\tau_W^1,\tau_W^2,\tau_W^3)$ is S−compact, then f is T−homomorphism.

Proof

(1) Since f is T −continuous, onto and one-to-one map, it is sufficient to show that f is T −closed. Let B be a τ_W^i−closed proper subset of a S −Lindelöf space $(W,\tau_W^1,\tau_W^2,\tau_W^3)$ .Then, by theorem [2.5] B is τ_W^j −Lindelöf, where $(i,j=1,2,3,i\neq j)$, Hence $f(B)$ is τ_Z^j −Lindelöf because $f:(W,\tau_W^j)\to(Z,\tau_Z^j)$ is continuous. By theorem [2.10] $f(B)$ is τ_Z^i −closed, where $(i,j=1,2,3)$. Hence $f:(W,\tau_W^i)\to(Z,\tau_Z^i)$ is closed for each $i,j=1,2,3$, so f is T−closed.

(2) The proof is similar to that in the statement **(1)**.

3 Reduce a tri-topological space to a single topology

In this section, we will introduce and discuss the necessary conditions so that the tri-topological space is reduced to a single topology.

Dear reader, here are some facts that will be used in this section.

(1) Every Lindelöf subset of a Hausdörff P-space is closed.

(2) Every compact subset of a Hausdörff space is closed.

Theorem 3.1 Let $(W,\tau_W^1,\tau_W^2,\tau_W^3)$ be a Hausdörff, S−Lindelöf, P−space, then $\tau_W^1=\tau_W^2=\tau_W^3$.

Proof First: we show that $\tau_W^1=\tau_W^2$. Let $\phi\neq u\in\tau_W^1$. Then $W-u$ is τ_W^1 −closed proper subset of a S−Lindelöf space W. By theorem [2.5] $W-u$ is τ_W^2 −Lindelöf. The fact " every τ_W^i −Lindelöf subset of a τ_W^i −Hausdörff P−space is τ_W^i −closed" Thus $W-u$ is τ_W^2 −closed. So, $u\in\tau_W^2$. Hence $\tau_W^1\subseteq\tau_W^2$. Similarly, we can show that $\tau_W^2\subseteq\tau_W^1$. Using the same technique we can show that $\tau_W^1=\tau_W^3$and $\tau_W^2=\tau_W^3$.

Theorem 3.2 Let $(W,\tau_W^1,\tau_W^2,\tau_W^3)$ be a Hausdörff, S−compact space, then $\tau_W^1=\tau_W^2=\tau_W^3$.

Proof First: we show that $\tau_W^1=\tau_W^2$. Let $\phi\neq u\in\tau_W^1$. Then $W-u$ is τ_W^1 −closed proper subset of a S−compact space W. By corollary [2.2] $W-u$ is τ_W^2 −compact. The fact " every τ_W^i-a compact subset of a τ_W^i −Housdörf space is τ_W^i −closed" Thus $W-u$ is τ_W^2 −closed. So, $u\in\tau_W^2$. Hence $\tau_W^1\subseteq\tau_W^2$. Similarly, we can show that $\tau_W^2\subseteq\tau_W^1$. Using the same technique we can show that $\tau_W^1=\tau_W^3$and $\tau_W^2=\tau_W^3$.

Theorem 3.3 Let $(W,\tau_W^1,\tau_W^2,\tau_W^3)$ be a Lindelöf T−Housdörf, P− space, then $\tau_W^1=\tau_W^2=\tau_W^3$.

Proof Let $u\in\tau_W^i$. Then $W-u$ is τ_W^i −closed proper subset of a Lindelöf P −space (W,τ_W^i), so $W-u$ is τ_W^i −Lindelöf. By theorem [2.10] $W-u$ is τ_W^j−closed ; $where\ (i\neq j: i,j=1,2,3)$.Hence u is τ_W^j−open. This implies that $\tau_W^i\subseteq\tau_W^j$. Similarly, we can show that $\tau_W^j\subseteq\tau_W^i$ where $(i\neq j: i,j=1,2,3)$. Thus; $\tau_W^1=\tau_W^2=\tau_W^3$.

Corollary 3.1 Let $(W,\tau_W^1,\tau_W^2,\tau_W^3)$ be a compact T−Housdörf, space, then $\tau_W^1=\tau_W^2=\tau_W^3$.

Proof Let $u\in\tau_W^i$. Then $W-u$ is τ_W^i −closed proper subset of a compact space (W,τ_W^i) so $W-u$ is τ_W^i −compact. By corollary [2.3] $W-u$ is τ_W^j−closed; where $(i\neq j: i,j=1,2,3)$. Hence u is τ_W^j−open. This implies that $\tau_W^i\subseteq\tau_W^j$. Similarly, we can show that $\tau_W^j \subseteq \tau_W^i$ where $(i \neq j: i,j = 1,2,3)$. Thus; $\tau_W^1 = \tau_W^2 = \tau_W^3$.

Lemma 3.1 Let $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ be a S$-$compact space and C be a τ_W^i $-$closed set such that $Int_j(X - C \neq \phi$, then C is $\tau Wi-$compact *(i,j=1,2,3,i≠j)*

Proof Let $x \in Int_j(W - C)$, then there exists a τ_W^j $-$open set U such that $x \in U \subseteq W - C$. Let $\mathcal{V} = \{v_\alpha : \alpha \in \Delta\}$ be a τ_W^i $-$open cover for C. For each $x_i \in W - C$ there exists a τ_W^i $-$open set $U(x_i)$ such that $x \in U(x_i) \subseteq W - C$. Since $\{v_\alpha : \alpha \in \Delta\} \cup \{U\} \cup \{U(x_1), U(x_2), U(x_3), \ldots\}$ is a $\tau_W^1 \tau_W^2 \tau_W^3 - an$ open cover of a S$-$compact space W, then there exists a finite subsets $\Delta^* \subseteq \Delta$ and $\{x_1, x_2, x_3, \ldots, x_n\} \subseteq W - C$ such that $\{v_\alpha : \alpha \in \Delta^*\} \cup \{u\} \cup \{U(x_1), U(x_2), U(x_3), \ldots, U(x_n)\}$ is a cover of W. Since $U(x_i) \cap C = \phi$ for all i and $U \cap C = \phi$, then $(\bigcup_{i=1}^n (U(x_i))) \cap C = \phi$. Thus $\{v_\alpha : \alpha \in \Delta^*\}$ is a finite subcover of $\mathcal{V}$ for C. Hence C is τ_W^i $-$compact.

Theorem 3.4 Let $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ be a T$-$Hausdörff S-compact space. If $Int_j(V) \neq \phi$ for all $V \in \tau_W^i - \{W\}$, then $\tau_W^i \subseteq \tau_W^j$where $(i,j = 1,2,3, i \neq j)$.

Proof Let $V \in \tau_W^i - \{W\}$, then $W - V$ is τ_W^i $-$closed with $Int_j(V) \neq \phi$. By the above lemma, W-V is τ_W^i $-$compact. By corollary [2.3] $W - V$ is τ_W^j $-$closed. So $V \in \tau_W^j - \{W\}$. Thus $\tau_W^i \subseteq \tau_W^j$.

Corollary 3.2 Let $(W, \tau_W^1, \tau_W^2, \tau_W^3)$ be a T $-$Hausdörff S-compact space. If $Int_j(V) \neq \phi$ for all $V \in \tau_W^i - \{W\}$ and $Int_i(V) \neq \phi$ for all $V \in \tau_W^j - \{W\}$, then $\tau_W^i = \tau_W^j$ where $(i,j = 1,2,3, i \neq j)$.

4 Results

Among the most important main and general results of this research are the following :

1. A definition of open covers in tri-topological spaces has been developed.

2. The definition of various concepts in singular or bi-topological spaces was generalized and studied in tri-topological spaces, with various examples. The topological relationships between those concepts were also clarified.

3. The main subject of the study is a generalization of scientific results, theories, and facts in Lindelöf's topological spaces, whether singular or pairwise.

4. A generalization of known results and theories in Lindelöf spaces and pairwise Lindelöf spaces in tri-topological spaces.

5. The necessary conditions for transforming the tri-topological space into a single topological space have been studied and developed.

6. The effect and characteristics of some types of functions on tri-Lindlöf spaces were studied.

5 Conclusions

Dear reader, as we have noticed in our study through the introduction and through the content of this research, this study is an extension of what the previous studies have reached in the field of tri-topological spaces in their various subjects and this is evident in the introduction to this research. Also, all the results that we obtained are generalizations of theories and known results on the main subject of the study, which is Lindelöf topological spaces.

Here are the most important points of the results of this research work:

1. Formulating the following definitions in the tri-topological spaces: tri-Lindlöf spaces, tri-compact spaces, and tri-countably compact spaces. Also, we clarify the relationship between them.

2. Illustrative examples were provided for all the concepts and theories contained in the research, which clarify the rationale of the theories or show the relationship between these concepts.

3. Special types of functions were defined in tri-spaces and their impact on the concepts of the study, especially its main subject was studied.

4. The necessary and sufficient conditions were studied in the tri-topological spaces $(\mathrm{W}, \tau_\mathrm{W}^1, \tau_\mathrm{W}^2, \tau_\mathrm{W}^3)$ through which it is reduced into a single space so that $\tau_\mathrm{W}^1 = \tau_\mathrm{W}^2 = \tau_\mathrm{W}^3$.

6 Recommendations and possible future studies

1. Researchers recommended the need to study the covering properties of the various tri-topological spaces in order to reach further relationships between them, especially Tri-compact and Tri-countably compact spaces.

2. This study can be applied in quadrilateral, pentagonal, and other topological spaces in a manner similar to what was achieved in this study or other studies in the same field.

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

HAMZA QOQAZEH: presented the main idea of the research, provided the basic definition of the research, and enriched it with several theories and illustrative examples.

Ali A.Atoom: wrote the introduction of the research and added some theories and scientific facts to the research.

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Conflict Of Interests

The researchers state that no personal objectives are gained through publishing this paper and confirm the originality of the work. The main objective of the current paper is to contribute to scientific research in the field of general topology. Also, all researchers acknowledge and attest that this research is not taken from any other source and is not published, sent for publication, or accepted for publication in any journal.

Alternatively, in case of no conflicts of interest, the following text will be published:

There are no conflicts of interest to state related to the content of this paper.