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BETWEEN PAIRWISE $-\alpha-$ PERFECT FUNCTIONS AND PAIRWISE $-T-$ $\alpha-$ PERFECT FUNCTIONS

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Abstract. Many academics employ various structures to expand topological space, including the idea of topology, as a result of the importance of topological space in analysis and some applications. One of the most notable of the generalizations was the definition of perfect functions in bitopological spaces, which was presented by Ali.A.Atoom and H.Z.Hdeib. We propose the notion of $\alpha-$ pairwise perfect functions in bitopological spaces and define different types of this concept in this study. Pairwise $-T-$ $\alpha-$ perfect functions, pairwise $-\alpha-$irr-perfect functions, and pairwise $-T-$ $\alpha-$ irr-perfect functions, are all characterized in addition to pairwise $-\alpha-$perfect functions. We go through their primary characteristics and show how they interact. Finally, under these functions, we introduce the images and inverse images of certain bitopological features. About these concepts, some product theorems have been discovered.

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1. Introduction

Covering spaces are well known to serve a significant role in topology [8], [17], [18]. Many authors investigated the connections between compactness and other topological and analytical ideas after the concept of compactness was defined see [19]. Furthermore, the topologists provided a variety of compactness generalizations based on the types of covers, open sets, and subcovers. The debate over these covering spaces is still a fascinating topic in topology. Also, a prominent topic in research is the idea of delivering weaker and stronger forms of open sets. J.C. Kelly [11] first proposed the concept of bitopological spaces

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in 1963. There are two (arbitrary) topologies in such spaces. For detailed definitions and notations, the reader should consult [11]. Kelly also applied some of the usual separation axioms produced in a topological space to a bitopological space see [6]. Pairwise regular, pairwise Hausdorff and pairwise normal are examples of such extensions[9]. There are several publications dedicated to the study of bitopologies or pairs of topologies on the same set;the majority of them are concerned with the theory, but only a few with applications. In this study, we look at the concept of pairwise perfect function in bitopological spaces and present some results. A set G equipped with two topologies θ_1 and θ_2 is called a bitopological space and will be denoted by (G, θ_1, θ_2). A cover $\hat{Q}$ of the space (G, θ_1, θ_2) is called p-open cover (Fletcher et al., 1969))[8] if $\hat{Q} \subseteq (G, \theta_1, \theta_2)$ and $\hat{Q}$ contains at least one non-empty member of θ_1 and at least one non-empty member of θ_2. A space (G, θ_1, θ_2) is said to be pairwise compact (p-compact) (Fletcher et al., 1969) if every p-open cover of X has a finite subcover. A subset K of (G, θ) is called semi-open (Levine, 1963)) [17], if $K \subseteq Cl(IntK)$. The complement of a semi-open set is called semi-closed (Biswas, 1970) [8]. The semi-interior of K, denoted by $sInt(K)$, is the union of all semi-open subsets of K while the semi-closure of A, denoted by $sCl(K)$, is the intersection of all semi$-$closed supersets of K. It is well known that $sInt(K) = K \cap Cl(IntK)$ and $sCl(K) = K \cup Int(Cl\ K)$. If K is a subset of (G, θ_1, θ_2) then the topologies on K inherited from θ_1 and θ_2 will be denoted by θ_{1K} and θ_{2K} respectively. In 1965, O.Nja°stad [14] introduced the notion of $\alpha-$sets. Since then, a large number of topologists studied various properties of point set topology with the help of $\alpha-$sets we can find that in [1] , [2] , [7] , [10] , [12] , [16] . In 1985, utilizing α-sets, Maheswari et al.[12], [13] defined the notion of $\alpha-$compactness in spaces with single topology. In 1988, Noiri et al. [14], [15] obtained further properties of this kind of spaces.The notion pairwise compactness is current in the existing literature. Pairwise open cover defined by Fletcher et al. [8] is instrumental for the introduction of this concept. In like manner, defining pairwise $\alpha-$cover , we have introduced pairwise $\alpha-$compact (briefly pαc) spaces. Ali.A.Atoom and H.Z.Hdeib [3] defined the perfect functions in the bitopological spaces and gives many properties of them.The notion of pairwise -perfect functions in bitoplogical spaces is defined in this study. Pairwise $\alpha-$perfect functions and pairwise $T-$ $\alpha-$ perfect functions are described in considerable detail. Under these functions, we also look at the images and inverse images of specific bitopological features. Finally, certain product theorems relating to these concepts were discovered.

2. Preliminaries

Many notions of generic topology were expanded by mathematicians. Separation and countability axioms, compactness, connectedness, paracompactness, metric space, and perfect functions are some of these concepts. We introduce the context of our investigation by using the term spaces to refer to bitopological spaces throughout this publication. The typical perfect functions operations and relations such as union, intersection, and inclusion will be followed. First, we'll go through the essential definitions and results that will be used throughout this project.Then we go through some of these functions' features and give some instances.

Definition 2.1. [12] In (G, θ), $K \subset G$ is called an $\alpha-$set iff $K \subset Int(Cl(Int(K)))$.

Nja°stad [14] used the symbol θ^{α} to denote the family of all $\alpha-$set in G and showed that θ^{α} is a topology on G.

Definition 2.2. [13] The complement of an $\alpha-$set is called $\alpha-$set closed. The family of all $\alpha-$set closed sets in G is denoted by $\pounds(\theta^{\alpha})$.

Definition 2.3. [8] Let (G, θ_1, θ_2) be bitopological space. $K \subset G$ is termed bi-compact iff K is both θ_1-compact and θ_2-compact.

Definition 2.4. [11] A cover $\hat{Q}$ of (G, θ_1, θ_2) is called pairwise open if $\hat{Q} \subset \theta_1 \cup \theta_2$,for $i = 1, 2$, $\hat{Q} \cap \theta_i \subset \{\phi \neq K \subset G\}$. If every pairwise open cover of (G, θ_1, θ_2) has a finite subcover, then the space is called pairwise compact.

Definition 2.5. [11] A space (G, θ_1, θ_2) is called pairwise Hausdroff (pairwise -T_2) if for each two distinct points g and h in G, there are a θ_1-open set Q and a θ_2-open set W such that $g \in Q$, $h \in W$, and $Q \cap W = \phi$.

Definition 2.6. [18] A function $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ is called pairwise continuous, if $\Psi_1 : (G, \theta_1) \to (H, \epsilon_1)$ and $\Psi_2 : (G, \theta_2) \to (H, \epsilon_2)$ are continuous functions .

Definition 2.7. [18] In a space (G, θ_1, θ_2), θ_1 is said to be regular with respect to θ_2, if for each point g in G and each θ_1-closed set N such that $g \notin N$,there are a θ_1-open set Q and θ_2-open set W such that, $g \in Q$, $N \subseteq W$, $Q \cap W = \phi$.

Definition 2.8. [9] A bitopological space (G, θ_1, θ_2) is said to be pairwise normal ,if given a θ_1-closed set N and θ_2-closed set O with $N \cap O = \phi$,there exist a θ_2-open set Q, and θ_1-open set W, such that $N \subseteq Q, O \subseteq W, Q \cap W = \phi$.

Definition 2.9. [11] A function $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ is called pairwise closed (p-open), if $\Psi_1 : (G, \theta_1) \to (H, \epsilon_1)$ and $\Psi_2 : (G, \theta_2) \to (H, \epsilon_2)$ are closed (open) functions.

I.e., if $N_1$is closed in θ_1, then $\Psi_1(N_1)$ is closed in ϵ_1, and if N_2 is closed in θ_2, then $\Psi_2(N_2)$ is closed in ϵ_2.

Definition 2.10. [3] A function $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ is called pairwise perfect if f is pairwise continuous, pairwise closed, and for each $h \in H$, $f^{-1}(h)$ is pairwise compact.

Definition 2.11. [16] A cover Q of (G, θ_1, θ_2) is termed pairwise $\alpha-$cover if $Q \subset \theta_1^{\alpha} \;\cup\; \theta_2^{\alpha}$ and $\hat{Q} \cap \theta_r \subset \{K \neq \phi\}$, $r = 1, 2$.

Definition 2.12. [7] A bitopological space (G, θ_1, θ_2) is said to be pairwise $\alpha-$compact, simply p.αc. if each p.α cover of (G, θ_1, θ_2) has a finite subcover.

Definition 2.13. [13] A space (G, θ_1, θ_2) is called pairwise-$\alpha-$Hausdroff (pairwise-$\alpha - T_2$) if for each two distinct points g and h, there are a θ_1^{α}-open set Q and a θ_2^{α}-open set$W$ such that $g \in Q,\ h \in W$, and $Q \cap W = \phi$.

Theorem 2.14. [9] *If (G, θ_1, θ_2) is pairwise Hausdorff and pairwise compact, then it is pairwise regular.*

Theorem 2.15. [11] *If (G, θ_1, θ_2) is pairwise compact and either τ_1 is regular with respect to θ_2 or θ_2 is regular with respect to θ_1, then it is pairwise normal.*

Theorem 2.16. [9] *If (G, θ_1, θ_2) is pairwise Hausdorff and bi-compact, then $\theta_1 = \theta_2$.*

Theorem 2.17. [9] *If (G, θ_1, θ_2) is bi-Hausdorff and pairwise compact, then $\theta_1 = \theta_2$.*

3. Main Results

The following definitions and results will be used to establish a sufficient requirement for pairwise $\alpha-$perfect and pairwise $T-\alpha-$perfect; such as pairwise $\alpha-$open, pairwise $T-\alpha-$compact, pairwise $\alpha-$Lindelof, pairwise $\alpha-$continuous, pairwise $\alpha-$irresolute, pairwise $\alpha-$closed.

Definition 3.1. A family $\hat{K}$ of subsets of a bitopological space (G, θ_1, θ_2) is called $\theta_1, \theta_2 - \alpha-$open if $\hat{K} \subset \theta_1^{\alpha} \cup \theta_2^{\alpha}$. If in addition, $\hat{K} \cap \theta_1^{\alpha} \neq \phi$ and $\hat{K} \cap \theta_2^{\alpha} \neq \phi$,then $\hat{K}$ is called pairwise $\alpha-$open.(simply pairwise-$\alpha-$open).

Definition 3.2. A bitopological space (G, θ_1, θ_2) is said to be pairwise- $T - \alpha-$compact, simply pairwise.T-$\alpha-$compact. if each pairwise.α (resp. $\theta_1\theta_2 - \alpha-$open) cover of G has a finite subcover.

The following question is natural: Is pairwise.T-$\alpha-$compact space for the topologies equivalent to pairwise.α.compact.?

Every pairwise.T-$\alpha-$compact space is pairwise-$\alpha-$compact and we can easily show that the converse is not be true. In see (Example 2.9) in [13].

Definition 3.3. A space (G, θ_1, θ_2) is called pairwise $\alpha-$Lindelof, simply p-$\alpha-$Lindelof if each p-$\alpha-$open cover of X has a countable subcover. It is obvious that every p-$\alpha-$compact space is p-$\alpha-$Lindelof, but the converse is not be true. In [13] see (Example 2.16).

To give a sufficient requirement for pairwise $\alpha - T-$perfect, and pairwise-$\alpha-$ perfect, to be coincident, the following three definitions will be used:

Definition 3.4. A function $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ is called pairwise $\alpha-$closed , if $\Psi_1 : (G, \theta_1) \to (H, \epsilon_1)$ and $\Psi_2 : (G, \theta_2) \to (H, \epsilon_2)$ are $\alpha-$closed functions.

I.e.N_1 is $\alpha-$closed in θ_1^α, then $\Psi(N_1)$ is $\alpha-$ closed in θ_1^α, and if N_2 is $\alpha-$closed in θ_2^α, then $\Psi(N_2)$ is $\alpha-$ closed in θ_2^α.

Definition 3.5. A function $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ is said to be pairwise $\alpha-$continuous, if $\Psi^{-1}(W)$ is $\alpha-$open ($\alpha-$closed)set for each pairwise open (pairwise closed) set W in H.

Definition 3.6. A function $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ is said to be pairwise $\alpha-$irresolute, if $\Psi^{-1}(W)$ is $\alpha-$open ($\alpha-$closed)set for each pairwise $\alpha-$open (pairwise $\alpha-$closed) set W in H.

Starting off, let's define the key ideas of the paper.

Definition 3.7. A function $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ is called pairwise $\alpha-$perfect function, if $\Psi$is pairwise $\alpha-$continuous, pairwise $\alpha-$closed, and for each $h \in H$, $\Psi^{-1}(h)$ is pairwise $\alpha-$compact.

Definition 3.8. A function $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ is called pairwise $\alpha-T-$perfect function, if f is pairwise $\alpha-$continuous, pairwise $\alpha-$closed, and for each $h \in H$, $\Psi^{-1}(h)$ is pairwise $T-\alpha-$compact.

Definition 3.9. A function $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ is called pairwise $\alpha-$irresolute$-$ perfect, if $\Psi$is pairwise $\alpha-$continuous, pairwise $\alpha-$closed, and for each $h \in H$, $\Psi^{-1}(h)$ is pairwise $\alpha-$irresolute.

Naturally, the following query arises: For the topologies, are pairwise $\alpha-$perfect functions comparable to perfect functions?

The following theorem gives that every pairwise $\alpha-$perfect function is a pairwise perfect function.

Theorem 3.10. *If a function $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ is pairwise $\alpha-$perfect function then it is pairwise perfect function*

Proof. Since $\theta \subset \theta^\alpha$ for every topology θ, it follows that every pairwise-$\alpha-$compact space is pairwise compact. By definition of pairwise- $\alpha-$perfect, then it is pairwise perfect. $\square$

The following example shows that the converse of Theorem 3.10 need not be true.

Example 3.11. Let R be the real line with $\theta_1 = \{R\} \cup \{K \subset R : 1 \notin K\}$and $\theta_2 = \{R\} \cup \{K \subset R : 2 \notin K\}$. We assert that only $\alpha-$set containing 1 in (R, θ_1) is R. Hence, any $\alpha-$cover O of (R, θ_1) surely contains R. So,$\{R\}($ is a finite subcover of O so that (R, θ_1) is $\alpha-$compact. Pursuing similar reasoning, we see that (R, θ_2) is $\alpha-$compact. But (R, θ_1, θ_2) is not pairwise- α c . For, if we consider the family $Q = \{\{g\} : g \in R\backslash \{1\}\} \cup \{1\}$. Hence Q is a pairwise $\alpha-$cover for (R, θ_1, θ_2). But it has no finite subcover. So, (R, θ_1, θ_2) is not pairwise- $\alpha-$ compact. Hence the result.

The following question is natural: Are pairwise $\alpha-$perfect functions for the bitopologies equivalent to pairwise $T - \alpha-$perfect functions?

The following proposition follows directly by Theorem 3.10

Proposition 3.12. *If a function $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ is pairwise $T-$ $\alpha-$perfect, then it is pairwise $\alpha-$perfect.*

We can quickly show that the reverse is untrue. The notation [5] is clear to see.

In this section, we introduce and investigate pairwise $\alpha-$perfect functions in bitopological spaces, focusing on the relationship between pairwise $\alpha-$perfect functions of a bitopological spaces and pairwise $T-$ $\alpha-$perfect functions generated in this bitopological spaces, the relationship between homogeneity of pairwise $\alpha-$perfect functions and pairwise $T-$ $\alpha-$perfect functions generated by bitopological spaces.

The theorem that follows is crucial, because it will be utilized to prove the next fundamental result.

Theorem 3.13. *If $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ is a pairwise $\alpha-$perfect function, then for every pairwise $\alpha-$compact subset $S \subseteq H$, the inverse image $\Psi^{-1}(S)$ is a pairwise $\alpha-$compact.*

Proof. Let $\underset{\sim}{Q} = \{Q_\gamma : \gamma \in \Theta\}$ be a pairwise-$\alpha-$open cover of (G, θ_1, θ_2), since Ψ is a pairwise $\alpha-$perfect function, then $\forall h \in H$, $\Psi^{-1}(h)$ is pairwise $\alpha-$compact, there exists a finite subsets Θ_h, $\overset{*}{\Theta}_h$ of Θ, such that $\Psi^{-1}(h) \subseteq \bigcup\limits_{\gamma \in \Theta_h} \{W_\gamma : \gamma \in \Theta_h\} \bigcup \bigcup\limits_{\gamma \in \overset{*}{\Theta}_h} \{J_\gamma : \gamma \in \overset{*}{\Theta}_h\}$, also $\{W_\gamma : \gamma \in \Theta_h\}$ is θ_1^α- open, $\{J_\gamma : \gamma \in \overset{*}{\Theta}_h\}$ is θ_2^α-open. Let $P_h = H - \Psi(G - \bigcup\limits_{\gamma \in \Theta_h} W_\gamma)$ is a ϵ_1^α-open set containing h, and $P_h^* = H - \Psi(G - \bigcup\limits_{\gamma \in \overset{*}{\Theta}_h} J_\gamma)$ is a ϵ_2^α-open set containing h, where $\Psi^{-1}(P_h) \subseteq \bigcup\limits_{\beta \in \Lambda_y} W_\gamma$, $\Psi^{-1}(P_h^*) \subseteq \bigcup\limits_{\gamma \in \overset{*}{\Theta}_h} J_\gamma$. Let $\underset{\sim}{P} = \{P_h : h \in H\} \bigcup \{P_h^* : h \in H\}$ is a pairwise- $\alpha-$an open cover of H. $\underset{\sim}{P}$ is pairwise $\alpha-$an open cover of S . Since S is pairwise $\alpha-$ compact, $S \subseteq \bigcup_{i=1}^{n}(P_{h_i}) \cup \bigcup_{i=1}^{m}(P^*_{h_j})$. Thus, $\Psi^{-1}(S) \subseteq$ □

$\bigcup_{i=1}^{n}\Psi^{-1}(P_{h_i}) \bigcup \bigcup_{j=1}^{m}\Psi^{-1}(P^*_{h_j}) \subseteq \bigcup_{\gamma\in\Theta_h} W_\gamma \cup \bigcup_{\gamma\in\overset{*}{\Theta}_h} J_\gamma$. It's demeaning $\Psi^{-1}(S)$ is pairwise $\alpha-$ compact.

Theorem 3.14. *If* $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ *is a pairwise* $T-$ $\alpha-$ *perfect function, then for every pairwise* $\alpha-$*compact subset* $S \subseteq H$*, the inverse image* $\Psi^{-1}(S)$ *is a pairwise* $T-$ $\alpha-$*compact.*

Proof. By the same technique used in proving theTheorem 4.1. □

The following Remarks follows directly by Theorems 4.1 and 4.2.

Remark 3.1. (i) A pairwise $\alpha-$compact space is inverse invariant under pairwise $\alpha-$ perfect function.

Remark 3.2. (ii) A pairwise $T-$ $\alpha-$compact space is inverse invariant under pairwise $T-\alpha-$perfect function.

The authors establish in [1] that a pairwise perfect function is the result of the combination of two pairwise perfect functions. Natural questions include the following two:

a) Is the composition of two pairwise $T-\alpha-$ perfect functions is a pairwise $\alpha-$ perfect function?

b) Is the composition of two pairwise $T-\alpha-$ perfect functions is a pairwise $\alpha-$ perfect function?

The following example shows that the composition of two pairwise $\alpha-$ perfect functions need not be a pairwise $\alpha-$ perfect function and the composition of two pairwise $T-\alpha-$ perfect functions need not be a pairwise $T-\alpha-$ perfect function.

Example 3.15. Let $G = \{1, 2, 3\}$, $\theta_1 = \{G, \phi, \{1\}, \{1, 3\}\}$, $\theta_2 = \{G, \phi, \{2, 3\}\}$, $H = \{4, 5, 6\}$, $\epsilon_1 = \{H, \phi, \{4\}, \{5\}\, \{4, 5\}\}$, $\epsilon_2 = \{H, \phi, \{5\}\}$, $S = \{7, 8, 9\}$, $\eta_1 = \{S, \phi, \{7\}\}$, $\eta_2 = \{S, \phi, \{8, 9\}\}$.

Let $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$, $\Pi : (H, \epsilon_1, \epsilon_2) \to (S, \eta_1, \eta_2)$, defined $\Psi(1) = 4, \Psi(2) = 6, \Psi(3) = 5,$ $\Pi(4) = 7, \Pi(5) = 9,$ $\Pi(6) = 8$. It is obvious that Ψ, Π are pairwise $\alpha-$continuous, but $\Pi \circ \Psi$ is not pairwise $\alpha-$ continuous, since $(\Pi \circ \Psi)^{-1}(\{7, 9\}) = \{1, 3\}$, which is not pairwise $\alpha-$ open set in G. Hence the composition of two pairwise $\alpha-$perfect functions is not necessary pairwise $\alpha-$ perfect functions, and the composition of two pairwise $T-\alpha-$ perfect functions need not be a pairwise $T-\alpha-$ perfect function.

The following theorem is a sufficient condition for two functions to be combined.

Theorem 3.16. *If* $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ *is pairwise perfect function and* $\Pi : (H, \epsilon_1, \epsilon_2) \to (S, \eta_1, \eta_2)$, *is pairwise* $\alpha-$*perfect function, then* $\Pi \circ \Psi$ *is pairwise* $\alpha-$*perfect function .*

Proof. Let K be any $\eta_1^\alpha-$ open set in S since Π is a pairwise $\alpha-$perfect function, then $\Pi^{-1}(K)$ is $\epsilon_1^\alpha-$ open set in H. Since Ψ is a pairwise perfect function, then $\Psi^{-1}(\Pi^{-1}(K))$ is a $\epsilon_1^\alpha-$ open set in G. Similarly let L be any $\eta_2^\alpha-$ open set in S, $\Pi \circ \Psi$ is a pairwise $\alpha-$perfect function. □

Using a manner similar to that used in the demonstration of Theorem [4.5], the proof of the following corollary follows directly.

Corollary 3.17. *If* $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ *is pairwise perfect function and* $\Pi : (H, \epsilon_1, \epsilon_2) \to (S, \eta_1, \eta_2)$ *is pairwise* $\alpha - T$ *perfect function,* $\Pi \circ \Psi$*is pairwise* $\alpha - T$ *perfect function.*

Several key features and links between pairwise $\alpha-$continuous, pairwise $\alpha-$perfect function, and composition functions are summarized in the following theorems.

Theorem 3.18. *If the composition* $\Pi \circ \Psi$ *of the pairwise* $\alpha-$*continuous function,* $\Psi : (G, \theta_1, \theta_2) \overset{onto}{\to} (H, \epsilon_1, \epsilon_2)$,*and pairwise* $\alpha-$ *perfect function* $\Pi : (H, \epsilon_1, \epsilon_2) \overset{onto}{\to} (S, \eta_1, \eta_2)$ *is a pairwise* $\alpha-$*closed, then the function*
$\Pi : (H, \epsilon_1, \epsilon_2) \overset{onto}{\to} (S, \eta_1, \eta_2)$ *is pairwise* $\alpha-$ *closed.*

Proof. Let K be a $\epsilon_1^\alpha-$closed in H, then $\Psi^{-1}(K)$ is $\theta_1^\alpha-$ closed in G. Since $\Pi\circ\Psi$ is pairwise $\alpha-$ closed, then $\Pi(\Psi\Psi^{-1}(K))$ is $\eta_1^\alpha-$closed in S,i.e $\Pi(K)$ is $\eta_1^\alpha-$ closed in S. Simillary we can show that if L be a $\epsilon_2^\alpha-$ closed in H, then $\Pi(L)$ is $\epsilon_2^\alpha-$closed in S . Thus Π is a pairwise $\alpha-$closed function. □

Theorem 3.19. *If the composition* $\Pi \circ \Psi$ *of the pairwise* $\alpha-$*continuous funcion*

$\Psi : (G, \theta_1, \theta_2) \overset{onto}{\to} (H, \epsilon_1, \epsilon_2)$, and a pairwise perfect function $\Pi : (H, \epsilon_1, \epsilon_2) \overset{onto}{\to} (S, \eta_1, \eta_2)$ is pairwise $\alpha-$ perfect, then the function $\Pi : (H, \epsilon_1, \epsilon_2) \overset{onto}{\to} (S, \eta_1, \eta_2)$ is pairwise $\alpha-$ perfect.

Proof. For every $s \in S$, $\Pi^{-1}(s) = \Psi((\Pi \circ \Psi)^{-1}(s))$ is pairwise $\alpha-$ compact, by theorem 4.7 $\Pi \circ \Psi$ is pairwise $\alpha-$perfect. Since Π is pairwise $\alpha-$ closed by theorem 4.5, we get that Π is pairwise $\alpha-$ perfect. □

Theorem 3.20. *If* $\Psi : (G, \theta_1, \theta_2) \overset{onto}{\to} (H, \epsilon_1, \epsilon_2)$ *is a pairwise* $\alpha-$*closed function, then for any* $L \subset H$ *the restriction* $\Psi_L : \Psi^{-1}(L) \to L$ *is pairwise* $\alpha-$ *closed.*

Proof. Let $L \subset H$ .Consider the function $\Psi : (G, \theta_1) \rightarrow (H, \epsilon_1)$, let K be a $\theta_1^\alpha-$ closed. Then $\Psi_L\ (K \bigcap \Psi^{-1}(L)) = \Psi(L) \bigcap L$ is $\epsilon_1^\alpha-$ closed in L. Similarly, we can show that if K a $\theta_2^\alpha-$ closed, $\Psi_L(K \bigcap \Psi^{-1}(L)) = \Psi(K) \bigcap L$ is $\epsilon_2^\alpha-$ closed in L. Thus $\Psi_L : \Psi^{-1}(L) \rightarrow L$ is pairwise $\alpha-$ closed. □

Proposition 3.21. *i) If* $\Psi : (G, \theta_1, \theta_2) \overset{onto}{\rightarrow} (H, \epsilon_1, \epsilon_2)$ *is a pairwise* $T-\alpha-$*perfect function,*

then for any $L \subset H$ the restriction $\Psi_L : \Psi^{-1}(L) \rightarrow L$ is pairwise $T-\alpha-$ perfect .

Proposition 3.22. *ii) If* $\Psi : (G, \theta_1, \theta_2) \overset{onto}{\rightarrow} (H, \epsilon_1, \epsilon_2)$ *is a pairwise* $T-\alpha-$*perfect function,*

then for any $L \subset H$ the restriction $\Psi_L : \Psi^{-1}(L) \rightarrow L$ is pairwise $T-\alpha-$perfect function.

we introduce characterizations and investigate the relationship between (pairwise $\alpha-$Hausdorff, pairwise $\alpha-$regular space, pairwise $\alpha-$normal, pairwise $\alpha-$paracompact, pairwise $\alpha-$homeomorphism, pairwise $\alpha-$strongly function) and (pairwise $\alpha-$perfect functions, pairwise $T-\alpha-$ perfect functions) in bitopological spaces.

The following lemma may be proved using a similar strategy as Remark 3.1 in [5] and will be used in the proof of the next two theorems.

Lemma 3.23. *A bitopological space* (G, θ_1, θ_2) *is p.α.c. if and only if each proper* θ_r^α*-closed subset of* (G, θ_1, θ_2) *is* $\alpha-$*compact relative to* (G, θ_e^α)*,where* $r,\ e = 1, 2;\ r \neq e$.

Theorem 3.24. *If* $\Psi : (G, \theta_1, \theta_2) \overset{onto}{\rightarrow} (H, \epsilon_1, \epsilon_2)$ *is pairwise* $\alpha-$*perfect,*

where (G, θ_1, θ_2) is pairwise $\alpha-$compact, and $(H, \epsilon_1, \epsilon_2)$ is pairwise $\alpha-$ Hausdorff, then Ψ is pairwise $\alpha-$ closed.

Proof. If K is $\theta_1^\alpha-$ closed subset of (G, θ_1, θ_2), then it is $\theta_2^\alpha-$ compact, because (G, θ_1, θ_2) is pairwise $\alpha-$compact. Since Ψ is pairwise $\alpha-$continuous. $\Psi(K)$ is a $\epsilon_2^\alpha-$compact subset of $(H, \epsilon_1, \epsilon_2)$. Since $(H, \epsilon_1, \epsilon_2)$ is pairwise $\alpha-$ Hausdorff, then $\Psi(K)$ is a $\epsilon_1^\alpha-$ closed. Simillary if L is a $\theta_2^\alpha-$ closed subset of G, then $\Psi(L)$ is a $\epsilon_2^\alpha-$closed subset of $(H, \epsilon_1, \epsilon_2)$. □

The proof of the following corollaly, which are similar to theorems 5.2.

Corollary 3.25. *If* $\Psi : (G, \theta_1, \theta_2) \overset{onto}{\rightarrow} (H, \epsilon_1, \epsilon_2)$ *is pairwise* $T-\alpha-$*perfect, where* (G, θ_1, θ_2) *is pairwise* $T-\alpha-$*compact, and* $(H, \epsilon_1, \epsilon_2)$ *is pairwise* $\alpha-$*Hausdorff, then* Ψ *is pairwise* $\alpha-$ *closed.*

A sufficient condition for a pairwise $\alpha-$perfect and a pairwise $T-\alpha-$perfect will be given by the following definition.

Definition 3.26. A function $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ is called pairwise $\alpha-$homeomorphism if $\Psi$is pairwise $\alpha-$continuous, pairwise $\alpha-$closed(pairwise $\alpha-$open), and $\Psi$is a bijection.

Theorem 3.27. *Let $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ be a p-$\alpha-$continuous bijection function. If (G, θ_1, θ_2) is pairwise $\alpha-$Hausdorff space, and (G, θ_1, θ_2) is pairwise-$\alpha-$compact, then Ψ is pairwise $\alpha-$ homeomorphism function.*

Proof. It's enough to show that Ψ is pairwise $\alpha-$closed. Let A be a θ_r^α-closed proper subset of B, and hence A is proper θ_e^α- compact, for $r,\ e = 1, 2;\ r \neq e$, by using Theorem 5.2 and hence $\Psi(A)$ is a ϵ_j^α-compact, but $(H, \epsilon_1, \epsilon_2)$ is pairwise $\alpha-$ Hausdorff space, $\Psi(N)$ is ϵ_r^α- closed, it is mean Ψ is pairwise $\alpha-$ homeomorphism function. □

The importance of the following definition in developing the concept of strongly and weakly functions will be used in the proof of the next main theorem.

Definition 3.28. A function $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ is called pairwise $\alpha-$strongly function(pairwise $\alpha-$weakly function), if for every pairwise $\alpha-$open cover $\underset{\sim}{Q} = \{Q_\gamma : \gamma \in \Theta\}$, there exists pairwise $\alpha-$open cover $\underset{\sim}{W} = \{W_\kappa : \kappa \in \Delta\}$ of H, such that $\Psi^{-1}(W) \subseteq \bigcup Q_\beta : \gamma \in \Theta_1, \Theta_1 \subset \Theta$, finite$\}, \forall w \in W$.

Theorem 3.29. *Let $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ be a pairwise $\alpha-$strongly onto function, then (G, θ_1, θ_2) is pairwise $\alpha-$compact, if $(H, \epsilon_1, \epsilon_2)$ is so.*

Proof. Let $\underset{\sim}{Q} = \{Q_\gamma : \gamma \in \Theta\}$ be a pairwise $\alpha-$open cover (G, θ_1, θ_2). Since Ψ is pairwise $\alpha-$ strongly function there exists pairwise $\alpha-$open cover $\underset{\sim}{W} = \{W_\kappa : \kappa \in \Delta\}$of $(H, \epsilon_1, \epsilon_2)$, such that $\Psi^{-1}(W) \subseteq \bigcup Q_\beta : \gamma \in \Theta_1,\ \Theta_1 \subset \Theta$, finite$\}, \forall w \in W$, but $(H, \epsilon_1, \epsilon_2)$ is pairwise $\alpha-$compact, so there exists $\Theta_1 \subset \Theta$, where Θ_1 is finite, such that $Y = \bigcup\limits_{\kappa \in \Delta_1} W_\kappa$. Hence, $G = \bigcup\limits_{\kappa \in \Delta_1} \Psi^{-1}(W_\kappa)$ so each $\Psi^{-1}(W_\kappa)$ contains finite members of $\underset{\sim}{Q}$. Thus G is pairwise $\alpha-$ compact.

In the sequel, the following five definitions will be used in theorem [5.13]. □

Definition 3.30. If $\underset{\sim}{Q}$ and $\underset{\sim}{A}$ are pairwise $\alpha-$ open covers of the bitopological space (G, θ_1, θ_2), then $\underset{\sim}{Q}$ is called a parallel refinement of $\underset{\sim}{A}$, if each $Q \in Q \cap \theta_r^\alpha$is contained in some $W \in \underset{\sim}{A} \cap \theta_r^\alpha,\ r = 1, 2$.

Definition 3.31. If $\underset{\sim}{Q}$ and $\underset{\sim}{A}$ are $\theta_1\theta_2 - \alpha-$ open covers of the bitopological space (G, θ_1, θ_2), then $\underset{\sim}{Q}$ is called a refinement of $\underset{\sim}{A}$ if each $Q \in Q \cap \theta_r^\alpha$ is contained in some $W \in \underset{\sim}{A} \cap \theta_r^\alpha,\ r = 1, 2$.

Definition 3.32. A family $\underset{\sim}{K}$ of subsets of a space (G,θ) is locally finite in (G,θ^{α}) if for each $g \in G$ there exists a $\alpha-$ open set Q such that $q \in Q$ and Q intersects at most finitely many elements of $\underset{\sim}{K}$.

Definition 3.33. A bitopological space (G,θ_1,θ_2) is called pairwise $T-\alpha-$ paracompact, if each pairwise $\alpha-$ open cover of G has a pairwise locally finite $\theta_1\theta_2$-$\alpha-$ open refinement.

Definition 3.34. A bitopological space (G,θ_1,θ_2) is called pairwise $\alpha-$ paracompact, if each pairwise $\alpha-$ open cover of G has a pairwise $\alpha-$ locally finite pairwise $\alpha-$ open refinement.

The following theorem enumerates several key features and relationships relating to the concepts in Definition [5.8-5.9-5.10-5.11-5.12].

Theorem 3.35. *Let* $\Psi:(G,\theta_1,\theta_2)\to(H,\epsilon_1,\epsilon_2)$ *be a pairwise* $\alpha-$ *perfect function, and* $(H,\epsilon_1,\epsilon_2)$ *is a pairwise* $T-\alpha-$ *paracompact, then* (G,θ_1,θ_2) *is so.*

Proof. Let $\underset{\sim}{Q} = \{Q_\gamma{:}\gamma \in \Theta\}$ be a p-$\alpha-$open cover of (G,θ_1,θ_2), since Ψ is a pairwise $\alpha-$perfect function, then $\forall h \in H,\ \Psi^{-1}(h)$ is pairwise $\alpha-$compact, there exists a finite subsets $\Theta_h,\ \overset{*}{\Theta}_h$ of Θ, such that $\Psi^{-1}(h) \subseteq \bigcup\limits_{\gamma\in\Theta_h}\{W_\gamma : \gamma \in \Theta_y\}\bigcup \bigcup\limits_{\gamma\in\overset{*}{\Theta}_h}\{J_\gamma : \gamma \in \overset{*}{\Theta}_h\}$, where $\{W_\gamma : \gamma \in \Theta_h\}$ is θ_1^{α}- open, $\{J_\gamma : \gamma \in \overset{*}{\Theta}_h\}$ is θ_2^{α}-open. Let $P_h = H - \Psi(G - \bigcup\limits_{\gamma\in\Theta_h} W_\gamma)$ is a ϵ_1^{α}-open set containing h, and $P_h^{*} =\ H - \Psi(G - \bigcup\limits_{\gamma\in\overset{*}{\Theta}_h} J_\gamma)$ is a ϵ_2^{α}-open set containing h, where $\Psi^{-1}(P_h) \subseteq \bigcup\limits_{\beta\in\Lambda_y} W_\gamma,\ \Psi^{-1}\ (P_h^{*}) \subseteq \bigcup\limits_{\gamma\in\overset{*}{\Theta}_h} J_\gamma$. Let $\underset{\sim}{P} = \{P_h : h \in\ H\ \}\bigcup\{P_h^{*} : h \in\ H\ \}$ is a pairwise $\alpha-$an open cover of H. $\underset{\sim}{P}$ is pairwise $\alpha-$an open cover of S. Since $(H,\epsilon_1,\epsilon_2)$ is pairwise $T-$ $\alpha-$ paracompact $\underset{\sim}{P}$ has a pairwise locally finite $\theta_1\theta_2-$ $\alpha-$open refinement. say $\underset{\sim}{M} = \{M_L : L \in \Delta_1\ \}\bigcup\{M_L^{*} : L \in \Delta_2\ \}$, □

where $\{M_L : L \in \Delta_1\ \}$ is ϵ_1^{α}-locally finite paracompact of P_h and $\{M_L^{*} : L \in \Delta_2\ \}$ is ϵ_2^{α}- locally finite paracompact of P_h^{*}, $\Delta = \Delta_1 \bigcup\ \Delta_2$. Let $U_1 = \{\Psi^{-1}(M_L)\ \bigcap W_{\gamma r},\ r = 1,2,...,\ n,\ L \in \Delta_1,\ \gamma \in \Theta_y\}$ is $\theta_1^{\alpha}-$open locally finite refinement of $\{W_\gamma : \gamma \in \Theta_y\}$, and let $U_2 = \{\Psi^{-1}\ (M_L^{*})\ \bigcap J_{\gamma_r},\ r = 1,2,...,n,\ L \in \Delta_2,\ \gamma \in \overset{*}{\Theta}_h\}$ is θ_2^{α}-open locally finite refinement of $\{J_\gamma : \gamma \in \overset{*}{\Theta}_h\}$.

Let $\underset{\sim}{U} = \{U_1 \bigcup U_2\}$, then $\underset{\sim}{U}$ is pairwise $\alpha-$ locally finite $\theta_1\theta_2-\alpha-$open refinement. $\underset{\sim}{Q}$, so (G,θ_1,θ_2) is a pairwise $T-\alpha-$ paracompact space.

The proof of the following corollaly, which are similar to Theorem 5.13.

Corollary 3.36. *Let* $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ *be a pairwise* $\alpha-$*perfect function ,*

and $(H, \epsilon_1, \epsilon_2)$ is a pairwise $\alpha-$paracompact, then$(G, \theta_1, \theta_2)$is so.

The following theorem enumerates several key features and relationships between pairwise $\alpha-$Hausdorff space and pairwise $\alpha-$perfect functions.

Theorem 3.37. *The pairwise* $\alpha-$*Hausdorff space is invariant under pairwise* $\alpha-$*perfect functions.*

Proof. Let (G, θ_1, θ_2) be a pairwise $\alpha-$ Hausdorff space, $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ be a pairwise $\alpha-$perfect function, and $h_1 \neq h_2$ in $(H, \epsilon_1, \epsilon_2)$, then $\Psi^{-1}(h_1)$, $\Psi^{-1}(h_2)$ are disjoint and pairwise $\alpha-$ compact subset of (G, θ_1, θ_2) . Since (G, θ_1, θ_2) be a p-$\alpha-$Hausdorff space, there exists a $\theta_1^\alpha-$neighborhood Q of G, and $\theta_2^\alpha-$neighborhood W, such that $\Psi^{-1}(h_1) \subseteq U$, $\Psi^{-1}(h_2) \subseteq V$, $Q \bigcap W = \phi$. Let the sets $H - \Psi(G-Q)$ be ϵ_1^α-open set in $(H, \epsilon_1, \epsilon_2)$ and containing h_1, $H - \Psi(G - W)$ be ϵ_2^α-open set in $(H, \epsilon_1, \epsilon_2)$ and containing h_2, such that $[H - \Psi(G - Q) \bigcap H - \Psi(G - W)] = H - [\Psi(G-Q) \bigcup \Psi(G-W)] = H - \Psi(G - Q \bigcap W) = H - \Psi(g) = \phi$. Hence $(H, \epsilon_1, \epsilon_2)$ is pairwise $\alpha-$Hausdorff space. □

Now, based on Theorem 5.15, we can make the following remarks:

(i) The pairwise $\alpha-$Hausdorff space is inverse invariant under pairwise $T-\alpha-$perfect.

(ii) The pairwise $\alpha-$Hausdorff space is inverse invariant under pairwise $\alpha-$perfect.

(iii) The pairwise $\alpha-$Hausdorff space is invariant under pairwise $T-\alpha-$perfect.

The following definition will be used to give a sufficient condition for lemma 5.17.

Definition 3.38. In a bitopological space (G, θ_1, θ_2), θ_1^α is said to be $\alpha-$regular with respect to θ_2^α if for each point g in G and each θ_1^α- closed set N such that $g \notin N$, there is a θ_1^α-open set Q and θ_2^α- open set W such that $g \in Q$, $N \subseteq W$ and $Q \cap W = \phi$. (G, θ_1, θ_2) is p -$\alpha-$regular if θ_1^α is $\alpha-$regular with respect to θ_2^α and vice versa.

The following lemma will be used in the proof of the next main theorem.

Lemma 3.39. *Let* G *be a pairwise* $\alpha-$*regular space, and* K *be* $\theta_r - \alpha-$*compact subset of* G, $r = 1, 2$, *then for each* $\theta_r - \alpha-$ *neighborhood* Q *of* K,

there exists a $\theta_r^\alpha-$open J, such that $K \subset J \subset s\ Cl_{\ \tau_e}(J) \subset Q$, r, $e = 1, 2$, $r \notin e$.

Proof. For each $k \in K$, there exist a $\theta_r^\alpha-$neighborhood $W(k)$ such that $s\ Cl_{\ \theta_e} W(k) \subset Q$, so $K \subset \bigcup_{i=1}^{n} W(k_i) \subset s\ Cl_{\ \theta_e} \bigcup_{i=1}^{n} W(k_i)$. Let $J = \bigcup_{k=1}^{n} W(k_i)$, then J is $\theta_r^\alpha-$open, but $s\ Cl_{\ \theta_e} J = s\ Cl_{\ \theta_e} \bigcup_{i=1}^{n} W(k_i) = s\ Cl_{\ \theta_e} \cup W(k_i)$. Hence $K \subset J \subset s\ Cl_{\ \theta_e} J \subset Q$, r, $e = 1, 2$; $r \neq e$ □

Theorem 3.40. *Let* $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ *be a pairwise* $\alpha-$*perfect function, and* (G, θ_1, θ_2) *is a pairwise* $\alpha-$*regular, then* $(H, \epsilon_1, \epsilon_2)$ *is so.*

Proof. Given ϵ_i^{α}-open set J, $h \in g$, r, $e = 1, 2$, $\Psi^{-1}(h) \in \Psi^{-1}(w)$ in H, since G is pairwise $\alpha-$regular, there exists $\theta_r^{\alpha}-$open set U, (by using Lemma 5.17), such that $\Psi^{-1}(h) \in s\ Cl\ _{\theta_e} \bigcup_{i=1}^{n} Q \subset \Psi^{-1}(w)$. Since Ψ is θ_r^{α}, then there exists $\epsilon_r^{\alpha}-$neighborhood J of h, such that $\Psi^{-1}(h) \in f^{-1}(j) \subset W$, but $J \subset \Psi(s\ Cl\ _{\tau_j} Q) \subset W$, since $\Psi(s\ Cl\ _{\tau_j} Q)$ is $\epsilon_r^{\alpha}-$closed, $h \in J \subset (s\ Cl\ _{\sigma_j}(J)) \subset \Psi(s\ Cl\ _{\tau_j} Q) \subset W$. Hence H is pairwise $\alpha-$ regular. □

Now, based on Theorem 5.18, we can make the following remarks:
a)The pairwise $\alpha-$regular space is inverse invariant under pairwise $\alpha-$perfect.
b)The pairwise $\alpha-$regular space is invariant under pairwise $T-\alpha-$perfect.
c) The pairwise $\alpha-$ regular space is inverse invariant under pairwise $T-\alpha-$perfect.

Proposition 3.41. *Let* $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ *be a pairwise perfect function, and* G *is pairwis* $\alpha-$*regular space then* H *is pairwis regular space.*

Proof. Since every $\alpha-$regular space is regular space (see [7]) , and by the definition of pairwise perfect function. Hence the result. □

In the sequel, the following definition will be used in Theorem 5.21.

Definition 3.42. A bitopological space (G, θ_1, θ_2) is called pairwise $\alpha-$normal, if each $\theta_r^{\alpha}-$closed set K and $\theta_e^{\alpha}-$closed set L, there exists $\theta_e^{\alpha}-$open set Q and $\theta_r^{\alpha}-$open set W, such that $K \subset Q,\ L \subset W,\ Q \cap W = \phi,\ r,\ e = 1,\ 2,\ r \neq e.$

Theorem 3.43. *Let* $\Psi : (G, \theta_1, \theta_2) \to (H, \epsilon_1, \epsilon_2)$ *be a pairwise* $\alpha-$*perfect function, and* (G, θ_1, θ_2) *is a pairwise* $\alpha-$*normal, then* $(H, \epsilon_1, \epsilon_2)$ *is so.*

Proof. It follows by using Lemma 5.17 and theorem 5.18. □

The following theorem and corollary show that some product and have been discovered.

Theorem 3.44. *Let* $(G, \theta_1, \theta_2), (H, \epsilon_1, \epsilon_2)$, *be any bitopological spaces. If* (G, θ_1, θ_2) *is pairwise* $T-\alpha-$*compact, then the projection function,* $\Phi : (G \times H,\ \theta_1 \times \epsilon_1\ , \theta_2 \times \epsilon_2) \to (H, \epsilon_1, \epsilon_2)$ *is pairwise* $\alpha-$*closed.*

Proof. If (G, θ_1, θ_2) is pairwise $T-\alpha-$compact, then (G, θ_1) is $T-\alpha-$compact, (G, θ_2) is $T-\alpha-$compact, thus the projection functions: $\Phi_1 : (G \times H,\ \theta_1 \times \epsilon_1 \ \to (H, \epsilon_1)$, $\Phi_2 : (G \times H,\ \theta_2 \times \epsilon_2 \ \to (H, \epsilon_2)$, are $\alpha-$ closed, thus Φ is pairwise $\alpha-$closed. □

Corollary 3.45. *Let* $(G, \theta_1, \theta_2), (H, \epsilon_1, \epsilon_2)$ *are pairwise* $T-\alpha-$*compact then* $(G \times H,\ \theta_1 \times \epsilon_1\ , \theta_2 \times \epsilon_2)$ *is pairwise* $T-\alpha-$*compact.*

4. Conclusions

In analysis, topology, and other fields, the usage of sets and functions for topological spaces has recently advanced dramatically. This study is primarily concerned with bitopological spaces. This work introduces a new class of sets, together with certain related functions and separation axioms, that can be used to define a variety of new topological spaces and functions. The study of pairwise $\alpha-$perfect functions and $T-\alpha-$perfect functions is of great importance because it provides a general frame that consists of parameterized classical bitopological spaces. The present work aims to study the perfect function of bitopological spaces in the $\alpha-$setting. Our results mainly investigate invariant properties between pairwise $\alpha-$perfect functions and pairwise $T-\alpha-$perfect functions. Thus, we define additive and finitely addictive properties. In this regard, we demonstrate some additive properties such as pairwise $\alpha-$open, pairwise $T-\alpha-$compact, pairwise $\alpha-$Lindelof, pairwise $\alpha-$continuous, pairwise $\alpha-$irresolute, pairwise $\alpha-$closed. With the help of illustrative examples. We show that the properties of pairwise $\alpha-$perfect functions and $T-\alpha-$pfuncerfect functions are not equivalent and focus on the homogeneity and relationship between pairwise $\alpha-$perfect functions and pairwise $T-\alpha-$perfect functions generated in this bitopological spaces. We intend to research more concepts in future studies, such as introducing characterizations and investigating the relationship between (pairwise $\alpha-$Hausdorff, pairwise $\alpha-$regular space, pairwise $\alpha-$normal, pairwise $\alpha-$paracompact, pairwise $\alpha-$homeomorphism, pairwise $\alpha-$strongly function) and (pairwise $\alpha-$perfect functions, pairwise $T-\alpha-$perfect functions) in bitopological spaces. Finally, our new findings are expected to have applications in general topology (particularly in characterizing some compactness notions) and a variety of other sciences, and we hope that this work will aid researchers interested in topological functions in studying properties as a new characteristic of the concepts. The results of this study can be further expanded according to the thought in [4, 5], which will be the way for much future research.

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