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STUDY THE STRUCTURE OF DIFFERENCE LINDELÖF TOPOLOGICAL SPACES AND THEIR PROPERTIES

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Abstract. In this paper, the concept of $D-$sets will be applied to create $D-$lindelöf spaces, a new type of topological space covering the property. This is performed by using a $D-$cover, which is a special type of cover. The primary purpose of this work is to introduce the principles and concepts of $D-$lindelöf spaces. We look into their properties as well as their relationships with other topological spaces. The basic relationship between $D-$lindelöf spaces and lindelöf spaces, as well as many other topological spaces, will be given and described, including $D-$compact, $D-$countably compact, and $D-$countably lindelöf spaces. Many novel theories, facts, and illustrative and counter-examples will be investigated. We will use several informative instances to explore certain of the features of the Cartesian product procedure across D-lindelöf spaces as well as additional spaces under more conditions.

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1. Introduction and introductory definitions

Open sets play a crucial role in defining several types of sets and various topological properties regarding these concepts in general topology. Tong [23] pioneered the study of $D-$sets in topology as a difference of two open sets subjected to additional criteria in 1982. He also introduces D_b spaces, which are a new sort of separation axiom. Recently, topological structures have had a lot of success in the field of general topology by developing new types of sets that are based on well-known types of sets. In our introduction, we give you, dear reader, an outline of these investigations as follows: Caldas cite [9] uses semi

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open sets, it is mean $sm-D_b$sets, to define semi $D-$sets in 1997. At that time, he develops a fresh set of separation axioms called $sm-D_b$ spaces. We find the connections between these separation axioms and the well-known D_b axioms. Since then, several authors have looked at the issue of how these new separation axioms are related. Hatir and Noiri [14] define C^*-D-sets as a distinction between C^*-sets in 1998. They use these sets to define C^*-D_b, C^*-D-compact space, as semi-open sets. In 2001, Jafari, [15] developed the idea of $p-$open sets and employed these definitions to produce $p-r-D-$sets. He explained the concept of $p-D_b$ spaces. The consequences of these axioms are examined. Caldas and Jafari, [10], created the idea of $D-$sets and proposed new separation axioms in 2002. The properties of these separations were investigated in depth. Caldas et. al. [11] presented crucial new concepts in 2003, based primarily on $\beta-$open sets dubbed $\beta-D_b$ $(b=0,1,2)$ spaces at the time. These separations' qualities were investigated. Following these key works on the subject, several research challenges attempting to study new types of continuous functions were discussed. $D-$sets, $D-SM-$sets, $D-$continuity, and $D-SM-$continuity were introduced by Ekici and Jafari [12] in 2008. After one year, Keskin and Noiri [17] presented the idea of $bD-$sets using $bD-$open sets. They use these ideas to formulate several axioms of weak separation. The implications of these novel separation axioms are acquired when compared to previous well-known axioms. As a new area in general topology, many interesting results have been presented and published. Through the study of $q-D_b$spaces, Balasubramanian [6] demonstrated the generalization of previously known separation axioms in 2010. Balasubramanian and Lakshim [7] define $q-p-r-D_b$ $(b=0,1,2)$ spaces in 2011 and go into great detail about them. In addition, in 2012, Jardo [16] introduced the notion of D_b-sets, which is based on its definition and research of topological features on $b-$open sets. D_b-spaces are among the separation axioms addressed. [20] Padma et al. The idea of G^*-D-sets was put up by P.P in 2017 and they employ G^*-open sets to create a new separation axiom, G^*-D_b $(b=0,1,2)$ spaces. The links between G^*-D_b $(b=0,1,2)$ sets and various forms of $D-$sets are investigated, including D, $p-r-D$, $sm-D$, $\beta-D$, $q-D$ and $q-p-r-$sets. Soft $(i\ ,\ j)^*$ $-$omega difference sets (briefly soft $(i\ ,\ j)^*-\hat{D}_w$-sets) and weak variants of soft $(i\ ,\ j)^*$ $-$omega difference sets were described by Sabiha and Abdul-Hady [18] in soft bitopological spaces in 2019. They studied the properties of several sorts of soft separation axioms, such as soft $(i\ ,\ j)^*-w-\hat{D}_k-$spaces, using these soft sets. In 2021, Qoqazeh et al. [21] established the concept of $D-$compact spaces. In this paper, the concept of $D-$sets will be applied to create $D-$lindelöf spaces, a new type of topological space covering property. This is performed by using a $D-$cover, which is a special type of cover. The primary purpose of this work is to introduce the principles and concepts of $D-$lindelöf spaces. We look into their properties as well as their relationships with other topological spaces.The basic relationship between $D-$lindelöf spaces and lindelöf spaces, as well as many other topological spaces, will be given and described, including $D-$compact, $D-$countably compact, and $D-$countably lindelöf spaces. Many novel theories, facts, and illustrative and counter-examples will be investigated. Using demonstrative instances, some properties of the cartesian product procedure among $D-$lindelöf spaces as well as additional spaces will be investigated with more conditions. Unless otherwise stated, (Q, γ) and (W, θ) or (Q and$W$) refer to topological spaces. $CL\,(L)$ and $Int\,(L)$ will be used to represent $\gamma-$closure and $\gamma-$interior of a set L, respectively. The product of γ_1 and γ_2 will be symbolized by the number $\gamma_1 \times \gamma_2$. You can look up some of the key terms we'll be using in this essay, like: $D-$set([9]), $D-$cover([21]), locally indiscrete ([10]), $(D_b, b = 0, 1, 2)$([22]), $D-$countably compact([21]), Lindelöf perfect([1]), [2], [3], [4], [5].

2. Between difference lindelöf spaces and difference compact spaces

In this part, the idea of $D-$lindelöf spaces in topological spaces is introduced, the differences between $D-$compact and $D-$lindelöf spaces are discussed, and provides examples and counter examples to illustrate the concepts.

Definition 2.1. *If every $D-$cover in a topological space (Q, γ) has a countable subcover, the space is said to be $D-$lindelöf.*

Definition 2.2. *The space is referred to as $D-$lindelöf, in the event that each $D-$cover in (Q, γ) has a countable subcover.*

Theorem 2.3. *Every $D-$lindelöf space is lindelöf space.*

Proof. The open cover of (Q, γ) is assumed to be $\breve{R} = \{R_\eta : \eta \in \Sigma\}$ $\breve{G}$ is a $D-$cover, that is, there is a countable subcover, which is another factor. □

The case that follows demonstrates that the previously stated theorem's reverse is not constantly true.

Example 2.4. *Let $Q = \mathbb{R}$,although (Q, γ_{cof}) is lindelöf, but it is not $D-$lindelöf. Any set of the type $Q - \{q\}\,, q \in \mathbb{R}$ is recognized as an open set in (Q, γ_{cof}). Let $G = Q - \{y\}$ and $H = Q - \{q\}$, then $G - H = \{q\}$, on the other hand, is a closed $D-$set. As a result, $\tilde{Q} = \{\{q\} : q \in \mathbb{R}\}$ is a $D-$cover of (Q, γ_{cof}), It does not have a countable subcover If $\tilde{Q} = \{\{q\} : q \in \mathbb{R}\}$has a countable subcover $\{\{q_1\}\ ,\ \{q_2\}\ ,\ ...\ ,\ \{q_n\}\,, ...\}$, we have $Q \subseteq \bigcup_{b=1}^{\infty} \{q_b\}$, Q is a subset of a set. Which is a contradiction.*

Corollary 2.5. *Every $D-$compact space is $D-$lindelöf space.*

Proof. We achieve the results in [21], and every $D-$lindelöf space is lindelöf space (theorem 2.2). □

The following instances demonstrate their relationship:

Example 2.6. *Let $Q = \mathbb{R}$ and $\gamma = \{\phi, Q, \{8\}, Q - \{8\}\}$. Hence, (Q, γ) is a $D-$lindelöf space since it is a $D-$compact space.*

Example 2.7. *Let $Q = \mathbb{R}$. Then (Q, γ_u) is not a $D-$compact space, and so is not $D-$lindelöf space. Since $\forall\ v \in \mathbb{N}$ Given $G_v = (-v\ , v)$ then the open cover $\breve{G} = \{G_v : v \in \mathbb{N}\}$ is a difference cover as well, despite not having a countable subcover.*

The contrary of the aforementioned theorem is demonstrated by the case that follows:

Example 2.8. *Let $Q = \mathbb{R}$. $(Q, \gamma_{l.r})$ is not lindelöf space, and as a result, it is not a $D-$lindelöf space.*

According to corollary 2.8, the opposite of the prior premise may be true under additional circumstances, as shown by the following theorem.

Corollary 2.9. *Each $D-$set is closed and open in a locally indiscrete space.*

Proof. Let $\grave{R} = G - H$ be two open sets. Allow closed sets G and H. When that happens, they become closed and open sets, like R since it represents the difference between two of them. □

Theorem 2.10. *Each locally indiscrete compact topological space (Q, γ) is $D-$lindelöf.*

Example 2.11. *Let $Q = \mathbb{R}$. Since it is lindelöf, it is obvious that the locally indiscrete (Q, τ_{ind}) is $D-$lindelöf.*

Example 2.12. *Let $Q = \mathbb{R}$ and $\gamma = \{\emptyset, Q, Q - \{7\}, \{7\}\}$. Then (Q, γ) is $D-$lindelöf.*

The following instances show that the converse is not ever the caes :

Example 2.13. *Suppose that Q be an infite set, then (Q, γ) is $D-$lindelöf space,*

and is not $D-$compact space.

Example 2.14. *Let γ_s denote Sorgenfrey line, then (Q, γ_s) is $D-$lindelöf space, and is not $D-$compact space.*

Example 2.15. *Let γ_{coc}, denote the cocountable topology, then $(\mathbb{Z}, \gamma_{coc})$ is $D-$lindelöf space, and is not $D-$compact space.*

Example 2.16. *Let $Q = \mathbb{R}, \gamma = \gamma_{coc}$, then any set of the form $\mathbb{R} - \{z_1, z_2\}$ is an open set, so $O_Z = \mathbb{R} - \{z_1, z_2\}, O_w = \mathbb{R} - \{w_1, w_2\}$,are open sets in γ_{coc}. Therefor, $O_Z - O_w = \{w_1, w_2\}$, or $O_Z - O_w = \{w_1\}$, or $O_Z - O_w = \{w_2\}$, then any set of the forms $\{\ q_1\}$, $\{q_1, q_2\}$ are $D-$sets in $(\mathbb{R}, \gamma_{coc})$, so it is $D-$lindelöf space, so that is lindelöf space, but $(\mathbb{R}, \gamma_{coc})$ is not $D-$compact and in addition it is not compact space.*

Example 2.17. *If Q is countable, then (Q, γ) is $D-$lindelöf space, then (N, γ_{dis}) is $D-$compact .*

3. Some properties of difference between Lindelöf Spaces

Within this section, we discuss some of the properties of $D-$lindelöf spaces as well as their relationships to other spaces, as well as many other topological spaces, will be given and described, including $D-$compact, $D-$countably compact, and $D-$countably lindelöf spaces.

Theorem 3.1. *For any topology on Q, (Q, γ) is a $D-$lindelöf space if Q is a countable set.*

Proof. Suppose $Q = \{q_1\ ,\ q_2\ ,\ ...\ ,\ q_n, ...\}$ be a countable set.
Let $\breve{R} = \{R_\eta : \eta \in \Sigma\}$ be a $D-$cover of Q. Now $\forall\ q_b \in Q$, choose $R_b \in \breve{R}$ such that $q_b \in R_b$. Consequently, a countable subcover of $\breve{R}$ for Q is $R^* = \{R_1\ ,\ R_2\ ,\ ....\}$Then Q is a $D-$lindelöf. □

Remark 3.2. *A $D-$set is any point at which two $D-$sets intersect.*

Proof. Suppose $R_1 = G_1 - H_1$ and $R_2 = G_2 - H_2$ be any two $D-$sets, then $R_1 \cap R_2 = (G_1 \cap G_2) - (H_1 \cup H_2)$ is a $D-$set. □

Using the principle of mathematical induction, it is simple to show the following corollary:

Remark 3.3. *A $D-$set need not be present in the union of any $D-$sets.*

Example 3.4. *Let $Q = \{l, j, k\}$ and $\gamma = \{\phi, Q,\ \{l\}, \{k\},\ \{l, k\}, \{j,\ k\}\}$. Then $R_1 = \{l\} = \{l, k\} - \{j\ , k\}$ and $R_2 = \{j\} = \{j\ , k\} - \{k\}$ are two $D-$sets. But $R_1 \cup R_2 = \{l\ ,\ j\}$is not a $D-$set. Since $\{l\ , j\} \neq G - H$ where G and H are two open sets and $Q \neq G$.*

Theorem 3.5. *If R_η is a differnce set in Q, then $R_\eta \cap L$ is a differnce set in (L, γ_L).*

Proof. Suppose R_η is a $D-$set in Q; in that case, $R_\eta = T - E$. T and E are open sets in Q and $T \neq Q$, respectively. $R_\eta \cap L = (T - E) \cap L = (T \cap L) - (E \cap L) = T_\eta - E_\eta$ is a $D-$set in (L, γ_L). Note that $T_\eta,\ E_\eta \in \gamma_L$. □

Theorem 3.6. *If (L, γ_L) is $D-$lindelöf, Afterwards there is a countable subcover for each and every cover of L.*

Proof. Presume that (L, γ_L) is a $D-$lindelöf space and that $\breve{R} = \{R_\eta : \eta \in \Sigma\}$ is a $D-$cover of L . $\forall\ \eta \in \Sigma$, $R^*_\eta = R_\eta \cap L$ is a difference sets in L, so that $\breve{R}^*_\eta = \left\{R^*_\eta : \eta \in \Sigma\right\}$ is a difference cover of L. $\breve{R}^*_\eta$ has a countable subcover for L in the form $\left\{R^*_{\eta_1}\ ,\ R^*_{\eta_2}\ ,\ ...\ ..\ \right\}$ because $\breve{R}\ \subset Q$ and (L, γ_L) is difference lindelöf. □

The following corollary can be established using the same procedures as in theorem3.8.

Corollary 3.7. *Let $L \subseteq (Q, \gamma)$. All of the covers for L by $D-$sets in Q must have a countable subcover in order for (L, γ_L) to be $D-$lindelöf .*

Theorem 3.8. *If O is a base for Q. If (Q, γ) is $D-$lindlöf, Consequently, each difference cover produced by $O\prime s$ elements has a countable subcover.*

Theorem 3.9. *Each $D-$cover produced by the elements of a base O of Q has a countable subcover if (Q, γ) is $D-$lindelöf.*

Proof. Take into account that is a $D-$lindelöf space, $\breve{R} = \{R_\eta : \eta \in \Sigma\}$ is a $D-$cover of Q produced by elements in O, and O is a base of Q, therefore it has a countable subcover. □

Theorem 3.10. *Let $\gamma_1 \subseteq \gamma_2$ and (Q, γ_1),(Q, γ_2) are topological spaces and . Given that (Q, γ_2) is $D-$lindelöf, (Q, γ_1) must also be.*

Proof. Assuming (Q, γ_1) is a $D-$cover, let $\breve{R} = \{R_\eta : \eta \in \Sigma\}$ Due to $\gamma_1 \subseteq \gamma_2$, $\breve{R}$ is likewise a $D-$cover of (Q, γ_2). A countable subcover exists as a result. □

Theorem 3.11. *A $D-$lindelöf space's closed subspaces are all $D-$lindelöf.*

Proof. L is its closed subset given a $D-$lindelöf space Q. Assume $\breve{R} \cup \{Q - L\}$is a $D-$cover of Q, A countable subcover $\breve{R}^*_\eta$ is therefore present because Q is $D-$lindelöf. Consequently, $\breve{R}^*_\eta - \{Q - L\}$ is a countable subcover of $\breve{R}$ for L. □

Theorem 3.12. *A $D-$lindelöf space's closed subspaces are all lindelöf .*

Proof. Given a $D-$lindelöf space Q, let L be its closed subset, $\breve{G} = \{G_\eta : \eta \in \Sigma\}$ is a cover for L, $\breve{G} \cup \{Q - L\}$ is a cover for Q. Consequently, $\breve{G}^* - \{Q - L\}$ is a countable subcover of $\breve{G}$ since Q is a $D-$lindelöf. □

Theorem 3.13. *Let (Q, γ) is D_2-space. If a $D-$lindelöf $C \subset Q$, is a locally indiscrete, then $\forall q \notin C$, $\exists\ T_q$, $T_w \ni q \in T_q$, $C \subseteq T_w$ and $T_q \cap T_w = \phi$.*

Proof. Assume $q \in Q - C$, and $\forall\ w \in C$,then $q \neq w$. Given that Q is a D_2-space, $\exists\ T_{1w}$, $T_{2w}\ \ni q \in T_{1w}$, $w \in T_{2w}$ and $T_{1w}\ \cap T_{2w}\ = \phi$. In the event that $\hat{T} = \{T_{2w} : w \in C\}$ is a $D-$cover of C. $\hat{T}$ has a countable subcover, $\breve{R}^*_\eta = \{R_{2w_1}\ , R_{2w_2}, ...\}$, because C is $D-$lindelöf. Assume $T_w = \bigcup\limits_{b=1}^{\infty} T_{2w_b}$ and $T_q = \bigcap\limits_{b=1}^{\infty} D_{1w_b}$, then T_q and T_w are $D-$sets $\ni q \in T_q$, $C \subseteq T_w$ and $T_q \cap T_w = \phi$. □

We shall give robust and significant outcomes for $D-$lindelöf spaces in this work, which will extend the reach of the previously discussed theory. In order to establish the connection between lindlöf and $D-$lindelöf , we prove a few theorems in Hausdorff and D_2 spaces.

Theorem 3.14. *Assume (Q, γ) is T_2-space and C is a $D-$lindelöf. If $C \subset Q$ then $\forall q \notin C$,$\exists$ an open sets G_q and H_q, $\ni q \in G_q$, $K \subseteq H_q$ and $G_q \cap H_q = \phi$.*

Proof. Let $q \in Q - C$, then $\forall\ w \in C$ we have $q \neq w$. Because Q is a T_2-space, $\exists$ an open sets G_w and $H_w \ni q \in G_w$, $w \in H_w$ and $G_w \cap H_w = \phi$. Assume $\breve{H} = \{H_w : w \in C\}$a cover of C, and so $\breve{H}$ is a $D-$cover of C. $\breve{H}$ has a countable subcover $\breve{H}^* = \{H_{b_1}, H_{b_2}, ...\}$ because C is $D-$lindelöf. Suppose $H_q = \bigcup\limits_{i=1}^{\infty} H_{w_b}$ and $G_q = \bigcap\limits_{b=1}^{\infty} G_{w_b}$, then H_q and G_q are open sets such that $q \in H_q$, $C \subseteq G_q$ and $H_q \cap G_q = \phi$. To clarify the previous fact; if $H_q \cap G_q \neq \phi$ then $H_q \cap G_{q_c} \neq \phi$ for $c \in \{1, 2, ...\}$. So $H_{q_c} \cap G_{q_c} \neq \phi$ since $H_q \subseteq H_{w_c}$, This is incongruous. $\square$

Theorem 3.15. *A T_2 space's $D-$lindlöf subsets are all closed.*

Proof. Assume (Q, γ) is T_2-space and C is a $D-$lindlöf. If $C \subset Q$, $q \in Q - C$, then based on a theorem[3.17], $\exists$ an open sets H_q and $G_q \ni q \in H_q$, $C \subseteq G_q$ and $H_q \cap G_q = \phi$. $q \in H_q \subseteq Q - G_q \subseteq Q - C$, so it's open. $\square$

We can demonstrate that in addition to locally indiscrete instead of T_2 space to any spaces and properties D_2-space to any subspaces using the same techniques as in theorem3.18.

4. Multiplication of D-lindelöf topology spaces

Many of the features of the Cartesian product procedure between D-lindelöf spaces along with additional spaces are examined in more detail using examples in this section.

Theorem 4.1. *D-lindelöf is a D-lindelöf space's continuous image.*

Theorem 4.2. *$D-$lindelöf is a continuous representation of a $D-$lindelöf space.*

Proof. Assume $F : Q \underset{continuous\ onto}{\rightarrow} W$ function and Q is a $D-$lindelöf space. Suppose $\breve{R}_w = \{R_\eta : \eta \in \Sigma\}$ be a $D-$cover of$W$. $\breve{R}_q = \left\{F^{-1}(R_\eta) : \eta \in \Sigma\right\}$ is a $D-$cover of Q. Because $\bigcup\limits_{\eta\in\Sigma} F^{-1}(R_\eta) = F^{-1}\left(\bigcup\limits_{\eta\in\Sigma} R_\eta\right) = F^{-1}(W) = Q$ and Q is $D-$lindelöf,$\left\{F^{-1}(R_{\eta_1}),\ F^{-1}(R_{\eta_2}), ...\right\}$ is a countable subcover in $\breve{R}_q$. Consequently, $\{R_{\eta_1}, R_{\eta_2}, ...\}$is a countable subcover of $\breve{R}_w$. $\square$

Definition 4.3. *If any $D-$set in $W\prime s$ inverse image is an open set in Q, then F is referred to as the $D-$irr function.*

Theorem 4.4. *If Q is lindelöf space, $F : Q \underset{D-irr\ function}{\longrightarrow} W$, then W is $D-$lindelöfspace*

Proof. Assume Q is a lindelöf space and $F : Q \underset{D-irr\ function}{\longrightarrow} W$ is onto function. If $\breve{R}_w = \{R_\eta : \eta \in \Sigma\}$ is a $D-$cover of W, $F : Q \underset{D-irr\ function}{\longrightarrow} W$, then $\breve{R}_q = \left\{F^{-1}(R_\eta) : \eta \in \Sigma\right\}$

is an open cover of Q, $\bigcup\limits_{\eta\in\Sigma} F^{-1}(R_\eta) = F^{-1}\left(\bigcup\limits_{\eta\in\Sigma} R_\eta\right) = F^{-1}(W) = Q$. Q is lindelöf, and $\breve{R}_w$ has a countable subcover as a result as $\{F^{-1}(R_{\eta_1}),\ F^{-1}(R_{\eta_2}),$ ....$\}$. Consequently,a countable subcover of $\breve{R}_w$ is $\{R_{\eta_1},\ R_{\eta_2}, ...\}$. $\square$

Theorem 4.5. *Assume Q is locally indiscrete space and $F : Q \underset{lindl\ddot{o}\ perfect}{\longrightarrow} W$ a function, If W is the case, then Q is a $D-$lindelöf space.*

Proof. Because F is perfect function, $\forall\ w \in W$, $\exists$ countable subsets η_w of η such that $F^{-1}(w) \subseteq \bigcup\limits_{\eta\in\Sigma} R_\eta$ is an open subset of Q, $Y_w = Y - F\left(Q - \bigcup\limits_{\eta\in\Sigma} R_\eta\right)$ $\underset{open}{\subset}$ W and $F^{-1}(Y_w) \subseteq \bigcup\limits_{\eta\in\Sigma} R_\eta$. Then $\grave{Y} = \{Y_w : w \in W\}$ is an open cover of W. Because W is $D-$lindelöf, $\grave{Y}$ has a countable subcover $\grave{Y}^* = \{Y_{w_b}\}_{b=1}^{\infty}$ it is mean $W = \bigcup\limits_{b=1}^{\infty} Y_{w_b}$. Consequently, $Q = F^{-1}(W) = F^{-1}\left(\bigcup\limits_{b=1}^{\infty} T_{w_b}\right) = \bigcup\limits_{i=1}^{\infty} F^{-1}(T_{w_b})$ and $F^{-1}(T_{w_b}) \subseteq \bigcup\limits_{\eta\in\Sigma} R_\eta$ and One or more countable members of R cover Q. $\square$

Theorem 4.6. *Assume that $F : Q \underset{lindl\ddot{o}\ perfect}{\longrightarrow} W$ is a function. Consequently, if W is $D-$lindelöf, Q is $D-$lindelöf.*

Proof. The proof is obtained by employing the same method as in the theorm 4.4. $\square$

Theorem 4.7. *If (Q,γ) is $D-$lindelöf, then $E_w : Q\times W \underset{projection}{\rightarrow} W$ is a perfect function.*

Proof. Because $E_w : Q\times W \underset{projection}{\rightarrow} W$ is a continuous function, Q is $D-$lindelöf, and $Q \times \{w\} \doteq Q$ .Thus, we deduce that $Q \times \{w\}$ is a $D-$lindelöf. $\forall\ w \in W$ we get $E_w^{-1}(w) = Q \times \{w\}$ is $D-$lindelöf, then it is lindelöf. $\square$

In the end, we demonstrate that E_w is closed. Suppose $w \in W$ and $E_w^{-1}(w) = Q \times \{w\} \subseteq G \underset{open}{\subset} Q \times W$. $\forall\ q \in Q$, $\exists$ an open sets H_{wq} and $G_q \ni q \in G_q$, $w \in H_{wq}$. Given that Q is a $D-$lindelöf space, $\breve{G}$ has a countable subcover with the notation $\{G_{qb}\}_{b=1}^{\infty}$. Consequently, $T_w = \bigcap\limits_{b=1}^{\infty} H_{wqb}$, then T_w is an open set containing w and $E_w^{-1}(T_w) = Q \times T_w \subseteq G$.

Theorem 4.8. *If $\prod\limits_{b=1}^{\infty} Q_b$ is $D-$lindelöf, then Q_b is $D-$lindelöf.* $\forall b = 1, 2, ...$

Proof. $\Longrightarrow$) Assume that $E_c : \prod_{b=1}^{\infty} X_b \underset{continuous\ onto}{\rightarrow} X_c$ function. Since $\prod_{b=1}^{\infty} X_b$ is $D-$lindelöf then X_c is $D-$lindelöf for all $b = 1, 2, ...$ □

Corollary 4.9. *If $\forall b = 1, 2, ...,$ Q_b is $D-$lindelöf, then $\prod_{b=1}^{\infty} Q_b$ is $D-$lindelöf.*

Proof. By employing the mathematical induction technique of evidence, it is easy to demonstrate this theoretical direction. □

Theorem 4.10. *If Q is $D-$lindelöf space, W is T_2 space and $F : Q \underset{continuos}{\rightarrow} W$, then F is closed*

Proof. Suppose $K \underset{closed}{\subset} Q$. So that K is $D-$lindelöf. Consequently, $F(K) \underset{closed}{\subset} W$. □

5. Conclusions and Future works

One of the most well-known extensions of topological spaces is lindelöf topology. In the context of lindelöf topology, we have dedicated this paper to presenting unique types of $D-$cover and separation axioms. These types were created using $D-$open sets. We looked at their most basic characteristics and discovered some connections between them. Some illustrated examples have been provided to validate the presented conclusions. In section 1, we review certain well-known definitions, hypotheses, and corollaries, as well as some key findings that will be used in the next section. In section2, explains how $D-$lindelöf spaces work by using the definition, compares $D-$compact and $D-$lindelöf spaces, and provides examples and counter examples to illustrate the concepts. In section 3, we discuss some of the properties of $D-$lindelöf spaces as well as their relationships to other spaces. In section 4, we investigate in further detail some of the properties of the outcome of the relationship between $D-$lindelöf spaces as well as additional spaces. including examples. We intend to examine the following issues to finish this study path: pairwise $D-$ compact space, pairwise $D-$ lindelöf, $D-$lindelöf perfect functions, $D-$locally compact, pairwise $D-$ countably compact, see [8], [22].

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