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Some New Concepts on Int-Soft Ideals in Ordered Semigroups

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In the present paper, we introduce some new notions on ordered semigroup. In fact, notion of a convex soft set in an ordered semigroup is introduced, and its basic properties are investigated. Moreover, we consider a characterization of a convex soft set. Furthermore, relations between a convex soft set and an int-soft l-ideal (or, int-soft r-ideal) are studied. Finally, int-soft l-ideals (or, int-soft r-ideals) generated by an ordered soft point are established.

Keywords: Soft set; int-soft l-ideal; int-soft r-ideal; ordered soft point.

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1. Introduction

The uncertainty which is appeared in economics, engineering, environmental science, medical science and social science, etc. is too complicated to be captured within a traditional mathematical framework. In order to overcome this situation, a number of approaches including fuzzy set theory,[32,33] probability theory, rough set theory,[29,30] vague set theory[7] and the interval mathematics[8] have been developed. Several tools such as fuzzy sets, rough sets and many others have been introduced to deal the uncertainty problems. Molodtsov[19] introduced a new mathematical tool soft sets to deal the problem of uncertainty. In soft sets the problems of uncertanties deals with enough numbers of parameters which makes it more accurate than other

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mathematical tools. Thus the soft sets are better than the other mathematical tools to describe the uncertanties. Molodtsov's soft set theory[19] is a kind of new mathematical model for coping with uncertainty from a parametrization point of view. In soft set theory, the problem of setting the membership function does not arise, which makes the theory easily applied to many different fields. The decision making problem in soft sets had been considered by Maji *et al.*.[17] In,[18] Maji *et al.* investigated several operations on soft sets. In,[31] Song initiated the study of int-soft semigroups, int-soft left (resp. right) ideals, and int-soft products. Furthermore, Jun *et al.*[14] introduced the concept of union-soft semigroups, union-soft l-ideals, union-soft r-ideals, and union-soft semiprime soft sets.

As a generalization of the concept of soft semigroups, soft left (right) ideals and left (right) idealistic soft semigroups, in 2010, Jun *et al.*[10] initiated the study of soft ordered semigroups, soft left (right) ideals and left (right) idealistic soft ordered semigroups. In recent years, a number of research papers have been devoted to the study of soft sets theory and its applications to the different algebraic structures (see for e.g.,[1–6,9–13,25–28,34]). Muhiuddin *et al.* have applied the fuzzy sets and related notions to the different aspects in semigroups theory (see for e.g., Refs. 20–24).

In this paper, we introduce the notion of a convex soft set in an ordered semigroup, and investigate its basic properties. Also, we consider a characterization of a convex soft set. Further, we discuss relations between a convex soft set and an int-soft l-ideal (or, int-soft r-ideal). Finally, we establish int-soft l-ideals (or, int-soft r-ideals) generated by an ordered soft point.

2. Preliminaries

An *ordered semigroup* (or, *po-semigroup*) is an ordered set $(T, \leq)$ which is a semigroup such that

$$(\forall a, b, x \in T)(a \leq b \Rightarrow xa \leq xb \text{ and } ax \leq bx). \tag{2.1}$$

An ordered semigroup T is said to be

- *left* (resp. *right*) *regular* if it satisfies:

$$(\forall\, a \in T)(\exists x \in T)(a \leq xa^2) \text{ (resp. } a \leq a^2x).$$

- *regular* if it satisfies:

$$(\forall\, a \in T)(\exists x \in T)(a \leq axa).$$

- *intra regular* if it satisfies:

$$(\forall\, a \in T)(\exists x, y \in T)(a \leq xa^2y).$$

For $A \subseteq T$, we denote

$$(A] := \{t \in T \mid t \leq h \text{ for some } h \in A\}.$$

For $A = \{a\}$, we write $(a]$ instead of $(\{a\}]$. For any $a \in T$, denote by $R(a)$ (resp. $L(a)$ and $I(a)$) the right (resp. left and two-sided) ideal of T generated by a. Note that $R(a) = (a \cup aT]$, $L(a) = (a \cup Ta]$ and $I(a) = (a \cup Ta \cup aT \cup TaT]$.

A nonempty subset I of an ordered semigroup T is called a *left* (resp. *right*) *ideal* of T if

$$TI \subseteq I \text{ (resp. } IT \subseteq I). \tag{2.2}$$

$$(\forall\, a, b \in T)(a \in I, b \leq a \Rightarrow b \in I). \tag{2.3}$$

In Ref. 19, Molodtsov have proposed soft set theory which provides a general mechanism for uncertainty modeling in a wide variety of applications. Let U be the universe of discourse and let E be the universe of all parameters related to the objects in U. A pair $(\tilde{\alpha}, A)$ is called a *soft set* over U, where $A \subseteq E$ and $\tilde{\alpha} : A \to \mathscr{P}(U)$ is a set-valued mapping, called the *approximate function* of the soft set $(\tilde{\alpha}, A)$ (see Ref. 19).

3. Some New Concepts on Int-Soft Ideals in Ordered Semigroups

In what follows, let T denote an ordered semigroup unless otherwise specified.

The *soft product*, denoted by $(\tilde{\alpha}, T) \,\tilde{\circ}\, (\tilde{\beta}, T)$, of two soft sets $(\tilde{\alpha}, T)$ and $(\tilde{\beta}, T)$ over U is defined to be the soft set $(\tilde{\alpha} \,\tilde{\circ}\, \tilde{\beta}, T)$ over U in which $\tilde{\alpha} \,\tilde{\circ}\, \tilde{\beta}$ is a mapping from T to $\mathscr{P}(U)$ given by

$$(\tilde{\alpha} \,\tilde{\circ}\, \tilde{\beta})(x) = \begin{cases} \displaystyle\bigcup_{(y,z) \in A_x} \{\tilde{\alpha}(y) \cap \tilde{\beta}(z)\} & \text{if } A_x \neq \emptyset, \\ \emptyset & \text{if } A_x = \emptyset, \end{cases}$$

where $A_x = \{(y, z) \in T \times T \mid x \leq yz\}$.

For any soft set $(\tilde{\alpha}, T)$ over U, consider a soft set $([\tilde{\alpha}], T)$ over U where

$$[\tilde{\alpha}] : T \to \mathscr{P}(U), \quad x \mapsto \bigcup_{x \leq y} \tilde{\alpha}(y).$$

Since $x \leq x$ for all $x \in T$, we have

$$[\tilde{\alpha}](x) = \bigcup_{x \leq y} \tilde{\alpha}(y) \supseteq \tilde{\alpha}(x)$$

for all $x \in T$. Hence $(\tilde{\alpha}, T) \,\tilde{\subseteq}\, ([\tilde{\alpha}], T)$.

A soft set $(\tilde{\alpha}, T)$ over U is said to be *convex* if $([\tilde{\alpha}], T) \,\tilde{\subseteq}\, (\tilde{\alpha}, T)$, and hence $(\tilde{\alpha}, T) = ([\tilde{\alpha}], T)$.

Theorem 3.1. *For a soft set $(\tilde{\alpha}, T)$ over U, the following are equivalent*:

(1) $(\tilde{\alpha}, T)$ *is convex.*
(2) $(\forall\, x, y \in T)(x \leq y \Rightarrow \tilde{\alpha}(x) \supseteq \tilde{\alpha}(y))$.

Proof. Assume that $(\tilde{\alpha}, T)$ is convex. Let $x, y \in T$ be such that $x \leq y$. Then

$$\tilde{\alpha}(x) = [\tilde{\alpha}](x) = \bigcup_{x \leq w} \tilde{\alpha}(w) \supseteq \tilde{\alpha}(y).$$

Conversely, if (2) is valid, then $[\tilde{\alpha}](x) = \bigcup_{x \leq y} \tilde{\alpha}(y) \subseteq \tilde{\alpha}(x)$ for all $x \in T$. Hence $([\tilde{\alpha}], S) \tilde{\subseteq} (\tilde{\alpha}, T)$, i.e., $(\tilde{\alpha}, T)$ is convex. □

Proposition 3.2. *For any soft sets $(\tilde{\alpha}, T)$, $(\tilde{\beta}, T)$ and $(\tilde{\gamma}, T)$ over U, we have*

(1) *If $(\tilde{\alpha}, T) \tilde{\subseteq} (\tilde{\beta}, T)$, then $([\tilde{\alpha}], T) \tilde{\subseteq} ([\tilde{\beta}], T)$.*
(2) $([\tilde{\alpha}], T) \tilde{\circ} ([\tilde{\beta}], T) \tilde{\subseteq} ([\tilde{\alpha} \tilde{\circ} \tilde{\beta}], T)$.
(3) $([\tilde{\alpha}], T)$ *is convex.*

Proof. (1) If $(\tilde{\alpha}, T) \tilde{\subseteq} (\tilde{\beta}, T)$, then $\tilde{\alpha}(x) \subseteq \tilde{\beta}(x)$ for all $x \in T$. Thus

$$[\tilde{\alpha}](x) = \bigcup_{x \leq y} \tilde{\alpha}(y) \subseteq \bigcup_{x \leq y} \tilde{\beta}(y) = [\tilde{\beta}](x)$$

for all $x \in T$. Therefore $([\tilde{\alpha}], T) \tilde{\subseteq} ([\tilde{\beta}], T)$.

(2) Let $x \in T$. If $A_x = \emptyset$, then $([\tilde{\alpha}] \tilde{\circ} [\tilde{\beta}])(x) = \emptyset \subseteq [\tilde{\alpha} \tilde{\circ} \tilde{\beta}](x)$. If $A_x \neq \emptyset$, then $x \leq yz$ for some $y, z \in T$. Thus

$$\begin{aligned} ([\tilde{\alpha}] \tilde{\circ} [\tilde{\beta}])(x) &= \bigcup_{(y,z) \in A_x} \{[\tilde{\alpha}](y) \cap [\tilde{\beta}](z)\} \\ &= \bigcup_{(y,z) \in A_x} \left\{ \left(\bigcup_{y \leq s} \tilde{\alpha}(s) \right) \cap \left(\bigcup_{z \leq t} \tilde{\beta}(t) \right) \right\} \\ &= \bigcup_{(y,z) \in A_x} \left\{ \bigcup_{y \leq s, z \leq t} \{\tilde{\alpha}(s) \cap \tilde{\beta}(t)\} \right\} \\ &\subseteq \bigcup_{(y,z) \in A_x} \left\{ \bigcup_{yz \leq st} \{\tilde{\alpha}(s) \cap \tilde{\beta}(t)\} \right\} \\ &= \bigcup_{(y,z) \in A_x} (\tilde{\alpha} \tilde{\circ} \tilde{\beta})(yz) \\ &= [\tilde{\alpha} \tilde{\circ} \tilde{\beta}](x). \end{aligned}$$

Therefore $([\tilde{\alpha}], T) \tilde{\circ} ([\tilde{\beta}], S) \tilde{\subseteq} ([\tilde{\alpha} \tilde{\circ} \tilde{\beta}], T)$.

(3) Let $x, y \in T$ be such that $x \leq y$. Then

$$[\tilde{\alpha}](y) = \bigcup_{y \leq z} \tilde{\alpha}(z) \subseteq \bigcup_{x \leq z} \tilde{\alpha}(z) = [\tilde{\alpha}](x).$$

It follows from Theorem 3.1 that $([\tilde{\alpha}], T)$ is convex. □

Let $(\tilde{\alpha}, T)$ be a soft set over U. For any $a \in T$ and $\lambda \in \mathscr{P}(U)\backslash\{\emptyset\}$, an *ordered soft point*, denoted by (a_λ, T), over U is defined to be a soft set over U where

$$a_\lambda : T \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \lambda & \text{if } x \in (a] \\ \emptyset & \text{otherwise} \end{cases}.$$

Denote by $a_\lambda \,\tilde{\in}\, \tilde{\alpha}$ we mean $(a_\lambda, T) \,\tilde{\subseteq}\, (\tilde{\alpha}, T)$, that is, $a_\lambda(x) \subseteq \tilde{\alpha}(x)$ for all $x \in T$.

Proposition 3.3. *If* $(\tilde{\alpha}, T)$ *is a convex soft set over* U, *then* $(\tilde{\alpha}, T) = (\tilde{\bigcup}_{a_\lambda \,\tilde{\in}\, \tilde{\alpha}} a_\lambda, T)$ *for any* $a \in T$ *and* $\lambda \in \mathscr{P}(U)\backslash\{\emptyset\}$.

Proof. Let (a_λ, T) be an ordered soft point over U such that $a_\lambda \,\tilde{\in}\, \tilde{\alpha}$. Then $a_\lambda(x) \subseteq \tilde{\alpha}(x)$ for all $x \in T$, and so

$$\left(\tilde{\bigcup_{a_\lambda \,\tilde{\in}\, \tilde{\alpha}}} a_\lambda\right)(x) = \bigcup_{a_\lambda \,\tilde{\in}\, \tilde{\alpha}} a_\lambda(x) \subseteq \bigcup_{a_\lambda \,\tilde{\in}\, \tilde{\alpha}} \tilde{\alpha}(x) = \tilde{\alpha}(x)$$

for all $x \in T$. Hence $(\tilde{\bigcup}_{a_\lambda \,\tilde{\in}\, \tilde{\alpha}} a_\lambda, T) \,\tilde{\subseteq}\, (\tilde{\alpha}, T)$. On the other hand, let $\tilde{\alpha}(x) = \lambda$ for $x \in T$. Then $x_\lambda \,\tilde{\in}\, \tilde{\alpha}$. In fact, if $y \notin (x]$ then $x_\lambda(y) = \emptyset \subseteq \tilde{\alpha}(y)$. If $y \in (x]$, then $y \le x$ and $x_\lambda(y) = \lambda$. Since $(\tilde{\alpha}, T)$ is convex, it follows from Theorem 3.1 that $\tilde{\alpha}(y) \supseteq \tilde{\alpha}(x) = \lambda = x_\lambda(y)$. Therefore $(x_\lambda, T) \,\tilde{\subseteq}\, (\tilde{\alpha}, T)$, that is, $x_\lambda \,\tilde{\in}\, \tilde{\alpha}$. Hence

$$\tilde{\alpha}(x) = \lambda = x_\lambda(x) \subseteq \left(\bigcup_{a_\lambda \,\tilde{\in}\, \tilde{\alpha}} a_\lambda\right)(x),$$

and so $(\tilde{\alpha}, T) \,\tilde{\subseteq}\, (\tilde{\bigcup}_{a_\lambda \,\tilde{\in}\, \tilde{\alpha}} a_\lambda, T)$. □

Definition 3.4. A soft set $(\tilde{\alpha}, T)$ over U is called an *intersection-soft l-ideal* (briefly, *int-soft l-ideal*) over U if it satisfies:

$$(\forall x, y \in T)(\tilde{\alpha}(xy) \supseteq \tilde{\alpha}(y)), \tag{3.1}$$

$$(\forall x, y \in T)(x \le y \Rightarrow \tilde{\alpha}(x) \supseteq \tilde{\alpha}(y)). \tag{3.2}$$

Definition 3.5. A soft set $(\tilde{\alpha}, T)$ over U is called an *intersection-soft r-ideal* (briefly, *int-soft r-ideal*) over U if it satisfies (3.2) and

$$(\forall x, y \in T)(\tilde{\alpha}(xy) \supseteq \tilde{\alpha}(x)). \tag{3.3}$$

If a soft set $(\tilde{\alpha}, T)$ over U is both an int-soft l-ideal and an int-soft r-ideal over U, we say that $(\tilde{\alpha}, T)$ is an *intersection-soft ideal* (briefly, *int-soft ideal*) over U.

Lemma 3.6. *For a soft set* $(\tilde{\alpha}, T)$ *over* U, *the following assertions are equivalent*:

(1) $(\tilde{\alpha}, T)$ *is an int-soft l-ideal* (*resp. int-soft r-ideal*) *over* U.
(2) $(\tilde{\alpha}, T)$ *satisfies the conditions* (3.2) *and*

$$(U_T, T) \,\tilde{\circ}\, (\tilde{\alpha}, T) \,\tilde{\subseteq}\, (\tilde{\alpha}, T) \qquad (\textit{resp. } (\tilde{\alpha}, T) \,\tilde{\circ}\, (U_T, T) \,\tilde{\subseteq}\, (\tilde{\alpha}, T)) \tag{3.4}$$

where (U_T, T) *is a soft set over* U *in which* $U_T(x) = U$ *for all* $x \in T$.

Theorem 3.7. *For any soft sets* $(\tilde{\alpha}, T)$, $(\tilde{\beta}, T)$ *and* $(\tilde{\gamma}, T)$ *over* U, *the following items are valid*:

(1) *If* $(\tilde{\alpha}, T)$ *is an int-soft ideal over* U, *then it is convex, i.e.,* $(\tilde{\alpha}, T) = ([\tilde{\alpha}], T)$.
(2) *If* $(\tilde{\alpha}, T)$ *and* $(\tilde{\beta}, T)$ *are int-soft* l*-ideals* (*resp. int-soft* r*-ideals*) *over* U, *then so are* $(\tilde{\alpha} \mathbin{\tilde{\circ}} \tilde{\beta}, T)$ *and* $(\tilde{\alpha} \mathbin{\tilde{\cup}} \tilde{\beta}, T)$.

Proof. (1) It follows from Theorem 3.1.
(2) Assume that $(\tilde{\alpha}, T)$ and $(\tilde{\beta}, T)$ are int-soft l-ideals over U. For any $x, y \in T$ with $x \leq y$, we have

$$(\tilde{\alpha} \mathbin{\tilde{\circ}} \tilde{\beta})(x) = \bigcup_{x \leq ab} \{\tilde{\alpha}(a) \cap \tilde{\beta}(b)\} \supseteq \bigcup_{y \leq ab} \{\tilde{\alpha}(a) \cap \tilde{\beta}(b)\} = (\tilde{\alpha} \mathbin{\tilde{\circ}} \tilde{\beta})(y).$$

Lemma 3.6 implies that

$$(U_T, T) \mathbin{\tilde{\circ}} ((\tilde{\alpha}, T) \mathbin{\tilde{\circ}} (\tilde{\beta}, T)) = ((U_T, T) \mathbin{\tilde{\circ}} (\tilde{\alpha}, T)) \mathbin{\tilde{\circ}} (\tilde{\beta}, T) \mathrel{\tilde{\subseteq}} (\tilde{\alpha}, T) \mathbin{\tilde{\circ}} (\tilde{\beta}, T).$$

It follows from Lemma 3.6 that $(\tilde{\alpha} \mathbin{\tilde{\circ}} \tilde{\beta}, T)$ is an int-soft l-ideal over U.

It is easy to verify that $(U_T, T) \mathbin{\tilde{\circ}} ((\tilde{\alpha}, T) \mathbin{\tilde{\cup}} (\tilde{\beta}, T)) \mathrel{\tilde{\subseteq}} (\tilde{\alpha}, T) \mathbin{\tilde{\cup}} (\tilde{\beta}, T)$. Let $x, y \in T$ be such that $x \leq y$. Then $\tilde{\alpha}(x) \supseteq \tilde{\alpha}(y)$ and $\tilde{\beta}(x) \supseteq \tilde{\beta}(y)$. Hence

$$(\tilde{\alpha} \mathbin{\tilde{\cup}} \tilde{\beta})(x) = \tilde{\alpha}(x) \cup \tilde{\beta}(x) \supseteq \tilde{\alpha}(y) \cup \tilde{\beta}(y) = (\tilde{\alpha} \mathbin{\tilde{\cup}} \tilde{\beta})(y).$$

Therefore $(\tilde{\alpha} \mathbin{\tilde{\cup}} \tilde{\beta}, T)$ is an int-soft l-ideal over U. Similarly, one can prove that $(\tilde{\alpha} \mathbin{\tilde{\circ}} \tilde{\beta}, T)$ and $(\tilde{\alpha} \mathbin{\tilde{\cup}} \tilde{\beta}, T)$ are int-soft r-ideals over U when $(\tilde{\alpha}, T)$ and $(\tilde{\beta}, T)$ are int-soft r-ideals over U. □

Theorem 3.8. *For any ordered soft point* (a_λ, T) *over* U *where* λ *is a nonempty subset of* U, *let* $(\tilde{\alpha}(a_\lambda), T)$ *be a soft set over* U *in which* $\tilde{\alpha}(a_\lambda)$ *is given as follows*:

$$\tilde{\alpha}(a_\lambda) : T \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \lambda & \textit{if } x \in L(a) \\ \emptyset & \textit{otherwise.} \end{cases}$$

Then $(\tilde{\alpha}(a_\lambda), T)$ *is an int-soft* l*-ideal over* U *generated by* (a_λ, T).

Proof. Let $(\tilde{\beta}, T)$ be a soft set over U with

$$\tilde{\beta} : T \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \lambda & \text{if } x \in L(a) \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $(\tilde{\beta}, T)$ is an int-soft l-ideal over U. In fact, let $x, y \in T$. If $\tilde{\beta}(y) = \emptyset$, then it is clear that $\tilde{\beta}(xy) \supseteq \tilde{\beta}(y)$. If $\tilde{\beta}(y) \neq \emptyset$, than $\tilde{\beta}(y) = \lambda$ and $y \in (S^1 a]$. Thus $y \leq ba$ for some $b \in T^1$, and so $xy \leq (xb)a$ and $xy \in L(a)$. It follows that $\tilde{\beta}(xy) = \lambda \supseteq \tilde{\beta}(y)$. Assume that $x \leq y$. If $y \notin L(a)$ then $\tilde{\beta}(x) \supseteq \emptyset = \tilde{\beta}(y)$. If $y \in L(a)$ then $x \in L(a)$ since $x \leq y$. Thus $\tilde{\beta}(x) = \lambda \supseteq \tilde{\beta}(y)$. Consequently, $(\tilde{\beta}, T)$ is an int-soft l-ideal over U. For each $x \in T$, if $x \in (a]$ then $x \in L(a)$ and so $a_\lambda(x) = \lambda = \tilde{\beta}(x)$. If $x \notin (a]$ then $a_\lambda(x) = \emptyset \subseteq \tilde{\beta}(x)$. Therefore $(a_\lambda, T) \mathrel{\tilde{\subseteq}} (\tilde{\beta}, T)$. Let $(\tilde{\gamma}, T)$ be an int-soft l-ideal over U such that $(a_\lambda, T) \tilde{\subseteq} (\tilde{\gamma}, T)$. If $y \in L(a)$, then there exists $b \in T^1$ such that $y \leq ba$. Hence

$$\tilde{\beta}(y) = \lambda = a_\lambda(a) \subseteq \tilde{\gamma}(a) \subseteq \tilde{\gamma}(ba) \subseteq \tilde{\gamma}(y).$$

If $y \notin L(a)$, then $\tilde{\beta}(y) = \emptyset \subseteq \tilde{\gamma}(y)$. Therefore $(\tilde{\beta}, T) \tilde{\subseteq} (\tilde{\gamma}, T)$. Consequently, $(\tilde{\beta}, T) = (\tilde{\alpha}(a_\lambda), T)$ is the int-soft l-ideal over U generated by (a_λ, T). □

Similarly, we have the following theorem.

Theorem 3.9. *For any ordered soft point (a_λ, T) over U where λ is a nonempty subset of U, let $(\tilde{\alpha}(a_\lambda), T)$ be a soft set over U in which $\tilde{\alpha}(a_\lambda)$ is given as follows*:

$$\tilde{\alpha}(a_\lambda) : T \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \lambda & \text{if } x \in R(a) \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $(\tilde{\alpha}(a_\lambda), T)$ is an int-soft r-ideal over U generated by (a_λ, T).

Theorem 3.10. *Let (a_λ, T) be an ordered soft points over U where λ is a nonempty subset of U. Then $(U_T, T) \tilde{\circ} (a_\lambda, T) \tilde{\circ} (U_T, T)$ is an int-soft ideal over U, and*

$$(U_T \tilde{\circ} a_\lambda \tilde{\circ} U_T)(x) = \begin{cases} \lambda & \text{if } x \in (TaT] \\ \emptyset & \text{if } x \notin (TaT] \end{cases}$$

for all $x \in T$.

Proof. Let $x \in T$. If $x \in (TaT]$, then there exist $y, z \in T$ such that $x \leq yaz$. Hence

$$\begin{aligned} (U_T \tilde{\circ} a_\lambda \tilde{\circ} U_T)(x) &= \bigcup_{x \leq x_1x_2x_3} \{U_T(x_1) \cap a_\lambda(x_2) \cap U_T(x_3)\} \\ &\supseteq U_T(y) \cap a_\lambda(a) \cap U_T(z) \\ &= U \cap \lambda \cap U = \lambda. \end{aligned}$$

On the other hand, $U_T(x_1) \cap a_\lambda(x_2) \cap U_T(x_3) = a_\lambda(x_2) \subseteq \lambda$ for all $x_1, x_2, x_3 \in T$, and so

$$(U_T \tilde{\circ} a_\lambda \tilde{\circ} U_T)(x) \subseteq \bigcup_{x \leq x_1x_2x_3} \{U_T(x_1) \cap a_\lambda(x_2) \cap U_T(x_3)\} \subseteq \lambda$$

for all $x \in T$. It follows that $(U_T \tilde{\circ} a_\lambda \tilde{\circ} U_T)(x) = \lambda$ for all $x \in (TaT]$. Assume that $x \notin (TaT]$. If there exist $x_1, x_2, x_3 \in T$ such that $x \leq x_1x_2x_3$, then

$$(U_T \tilde{\circ} a_\lambda \tilde{\circ} U_T)(x) = \bigcup_{x \leq x_1x_2x_3} \{U_T(x_1) \cap a_\lambda(x_2) \cap U_T(x_3)\} = \bigcup_{x \leq x_1x_2x_3} a_\lambda(x_2).$$

If $(U_T \tilde{\circ} a_\lambda \tilde{\circ} U_T)(x) \neq \emptyset$, then there exist $a, b, c \in T$ such that $x \leq abc$ and $b \in (a]$. Thus $x \in (TaT]$ which leads a contradiction. Therefore $(U_T \tilde{\circ} a_\lambda \tilde{\circ} U_T)(x) = \emptyset$. If there does not exist $x_1, x_2, x_3 \in T$ such that $x \leq x_1 x_2 x_3$, then $(U_T \,\tilde{\circ}\, a_\lambda \,\tilde{\circ}\, U_T)(x) = \emptyset$. Now, it is easy to verify that

$$(U_T, T) \,\tilde{\circ}\, (U_T \,\tilde{\circ}\, a_\lambda \,\tilde{\circ}\, U_T, T) \,\tilde{\subseteq}\, (U_T \,\tilde{\circ}\, a_\lambda \,\tilde{\circ}\, U_T, T)$$

and

$$(U_T \,\tilde{\circ}\, a_\lambda \,\tilde{\circ}\, U_T, T) \,\tilde{\circ}\, (U_T, T) \,\tilde{\subseteq}\, (U_T \,\tilde{\circ}\, a_\lambda \,\tilde{\circ}\, U_T, T).$$

Let $x, y \in T$ be such that $x \leq y$. Obviously, $(U_T \,\tilde{\circ}\, a_\lambda \,\tilde{\circ}\, U_T)(x) \subseteq \lambda$ for all $x \in T$. If $x \in (TaT]$, then $(U_T \,\tilde{\circ}\, a_\lambda \,\tilde{\circ}\, U_T)(x) = \lambda \supseteq (U_T \,\tilde{\circ}\, a_\lambda \,\tilde{\circ}\, U_T)(y)$. If $x \notin (TaT]$, then $y \notin (TaT]$ and so $(U_T \,\tilde{\circ}\, a_\lambda \,\tilde{\circ}\, U_T)(x) = \emptyset = (U_T \,\tilde{\circ}\, a_\lambda \,\tilde{\circ}\, U_T)(y)$. Therefore, using Lemma 3.6, we know that $(U_T, T) \,\tilde{\circ}\, (a_\lambda, T) \,\tilde{\circ}\, (U_T, T)$ is an int-soft ideal over U. □

Similarly, we have the following theorems.

Theorem 3.11. *Let (a_λ, T) be an ordered soft points over U where λ is a nonempty subset of U. Then $(U_T, T) \,\tilde{\circ}\, (a_\lambda, T)$ is an int-soft l-ideal over U, and*

$$(U_T \,\tilde{\circ}\, a_\lambda)(x) = \begin{cases} \lambda & \text{if } x \in (Ta] \\ \emptyset & \text{if } x \notin (Ta] \end{cases}$$

for all $x \in S$.

Theorem 3.12. *Let (a_λ, T) be an ordered soft points over U where λ is a nonempty subset of U. Then $(a_\lambda, T) \,\tilde{\circ}\, (U_T, T)$ is an int-soft r-ideal over U, and*

$$(a_\lambda \,\tilde{\circ}\, U_T)(x) = \begin{cases} \lambda & \text{if } x \in (aT] \\ \emptyset & \text{if } x \notin (aT] \end{cases}$$

for all $x \in T$.

Proposition 3.13. *For any nonempty subsets λ and δ of U, if a_λ and b_δ are ordered soft points over U, then $(a_\lambda \,\tilde{\circ}\, b_\delta, T) = ((ab)_{\lambda\cap\delta}, T)$.*

Proof. Let $x \in T$. If $x \in (ab]$, then

$$(a_\lambda \,\tilde{\circ}\, b_\delta)(x) = \bigcup_{(y,z)\in A_x} \{a_\lambda(y) \cap b_\delta(z)\} \supseteq a_\lambda(a) \cap b_\delta(b) = \lambda \cap \delta.$$

Note that $a_\lambda(y) \cap b_\delta(z) \subseteq \lambda \cap \delta$ for all $y, z \in T$. Hence $(a_\lambda \,\tilde{\circ}\, b_\delta)(x) \subseteq \lambda \cap \delta$. It follows that

$$(a_\lambda \,\tilde{\circ}\, b_\delta)(x) = \lambda \cap \delta = (ab)_{\lambda\cap\delta}(x).$$

For $x \notin (ab]$, assume that $(a_\lambda \,\tilde{\circ}\, b_\delta)(x) \neq \emptyset$. Then

$$(a_\lambda \,\tilde{\circ}\, b_\delta)(x) = \bigcup_{(y,z)\in A_x} \{a_\lambda(y) \cap b_\delta(z)\} \neq \emptyset,$$

and so $a_\lambda(y_0) \cap b_\delta(z_0) \neq \emptyset$ for some $y_0, z_0 \in T$ with $x \leq y_0 z_0$. Hence $y_0 \in (a]$ and $z_0 \in (b]$. It follows that $x \in y_0 z_0 \subseteq (a](b] \subseteq (ab]$, which is a contradiction. Therefore

$(a_\lambda \tilde{\circ} b_\delta)(x) = \emptyset = (ab)_{\lambda\cap\delta}(x)$. Consequently, we know that

$$(a_\lambda \tilde{\circ} b_\delta)(x) = (ab)_{\lambda\cap\delta}(x)$$

for all $x \in T$, that is, $(a_\lambda \tilde{\circ} b_\delta, T) = ((ab)_{\lambda\cap\delta}, T)$. $\square$

Corollary 3.14. *For any nonempty subsets λ and δ of U, if a_λ and b_δ are ordered soft points over U, then*

$$(a_\lambda \tilde{\circ} b_\delta, T) = (b_\delta \tilde{\circ} a_\lambda, T) \Leftrightarrow ab = ba.$$

Proof. It is straightforward. $\square$

Proposition 3.15. *For any ordered soft points a_λ and b_δ over U where λ and δ are nonempty subsetes of U, we have*

$$b_\delta \tilde{\in} U_T \tilde{\circ} a_\lambda \tilde{\circ} U_T \Leftrightarrow b \in (TaT], \quad \delta \subseteq \lambda.$$

Proof. If $b_\delta \tilde{\in} U_T \tilde{\circ} a_\lambda \tilde{\circ} U_T$, then $(U_T \tilde{\circ} a_\lambda \tilde{\circ} U_T)(b) \supseteq b_\delta(b) = \delta \neq \emptyset$. Hence

$$(U_T \tilde{\circ} a_\lambda \tilde{\circ} U_T)(b) = \lambda \supseteq \delta \quad \text{and} \quad b \in (TaT]$$

by Theorem 3.10.

Conversely, assume that $b \in (TaT]$ and $\delta \subseteq \lambda$. For any $x \in T$, if $x \in (b]$ then $x \in (b] \subseteq ((TaT]] = (TaT]$. It follows from Theorem 3.10 that $(U_T \tilde{\circ} a_\lambda \tilde{\circ} U_T)(x) = \lambda \supseteq \delta = b_\delta(x)$. If $x \notin (b]$, then $b_\delta(x) = \emptyset \subseteq (U_T \tilde{\circ} a_\lambda \tilde{\circ} U_T)(x)$. Therefore $b_\delta \tilde{\in} U_T \tilde{\circ} a_\lambda \tilde{\circ} U_T$. $\square$

For any subset D of T and a nonempty subset λ of U, let $(\tilde{\alpha}_D, T)$ and $((\lambda\tilde{\alpha})_D, T)$ be soft sets over U given as follows:

$$\tilde{\alpha}_D : T \to \mathscr{P}(U), \quad x \mapsto \begin{cases} U & \text{if } x \in D \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$(\lambda\tilde{\alpha})_D : T \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \lambda & \text{if } x \in D \\ \emptyset & \text{otherwise,} \end{cases}$$

respectively. Obviously, if $D = (a]$ then $(\lambda\tilde{\alpha})_D = a_\lambda$.

Proposition 3.16. *For any nonempty subsets D and E of T and any nonempty subset λ of U, we have the following assertions.*

(1) $((\lambda\tilde{\alpha})_D \tilde{\circ} (\lambda\tilde{\alpha})_E, T) = ((\lambda\tilde{\alpha})_{(DE]}, T)$.
(2) $((\lambda\tilde{\alpha})_D \tilde{\cap} (\lambda\tilde{\alpha})_E, T) = ((\lambda\tilde{\alpha})_{D\cap E}, T)$.
(3) $((\lambda\tilde{\alpha})_D \tilde{\cup} (\lambda\tilde{\alpha})_E, T) = ((\lambda\tilde{\alpha})_{D\cup E}, T)$.
(4) $((\lambda\tilde{\alpha})_{(D]}, T) = (\tilde{\bigcup}_{a\in D} a_\lambda, T)$.

Proof. (1) If $x \in (DE]$ then $\tilde{\alpha}_{(DE]}(x) = U$ and $x \leq ab$ for some $a \in D$ and $b \in E$. Hence $(a, b) \in A_x$ and thus

$$\begin{aligned} ((\lambda\tilde{\alpha})_D \,\tilde{\circ}\, (\lambda\tilde{\alpha})_E)(x) &= \bigcup_{(y,z)\in A_x} \{(\lambda\tilde{\alpha})_D(y) \cap (\lambda\tilde{\alpha})_E(z)\} \\ &\supseteq (\lambda\tilde{\alpha})_D(a) \cap (\lambda\tilde{\alpha})_E(b) \\ &= \lambda \cap \lambda = \lambda. \end{aligned}$$

Since $(\lambda\tilde{\alpha})_D(y) \subseteq \lambda$ and $(\lambda\tilde{\alpha})_E(z) \subseteq \lambda$ for all $y, z \in T$, we get $((\lambda\tilde{\alpha})_D \,\tilde{\circ}\, (\lambda\tilde{\alpha})_E)(x) \subseteq \lambda$. Therefore $((\lambda\tilde{\alpha})_D \,\tilde{\circ}\, (\lambda\tilde{\alpha})_E)(x) = \lambda = (\lambda\tilde{\alpha})_{(DE]}(x)$.

If $x \notin (DE]$ then $(\lambda\tilde{\alpha})_{(DE]}(x) = \emptyset$. For the case $A_x = \emptyset$, we have

$$((\lambda\tilde{\alpha})_D \,\tilde{\circ}\, (\lambda\tilde{\alpha})_E)(x) = \emptyset = (\lambda\tilde{\alpha})_{(DE]}(x).$$

The case $A_x \neq \emptyset$ implies that $x \leq yz$ for all $(y, z) \in A_x$. If $y \in D$ and $z \in E$, then $yz \in DE$ and so $x \in (DE]$. This is impossible, and thus $y \notin D$ or $z \notin E$. If $y \notin D$, then $(\lambda\tilde{\alpha})_D(y) = \emptyset$ and thus $(\lambda\tilde{\alpha})_D(y) \cap (\lambda\tilde{\alpha})_E(z) = \emptyset$. Similarly, if $z \notin E$ then $(\lambda\tilde{\alpha})_D(y) \cap (\lambda\tilde{\alpha})_E(z) = \emptyset$. Therefore

$$((\lambda\tilde{\alpha})_D \,\tilde{\circ}\, (\lambda\tilde{\alpha})_E)(x) = \bigcup_{(y,z)\in A_x} \{(\lambda\tilde{\alpha})_D(y) \cap (\lambda\tilde{\alpha})_E(z)\} = \emptyset.$$

Consequently, (1) is true.

(2) Let $x \in T$. If $x \in D \cap E$, then $x \in D$ and $x \in E$, and so

$$(\lambda\tilde{\alpha})_{D\cap E}(x) = \lambda = \lambda \cap \lambda = (\lambda\tilde{\alpha})_D(x) \cap (\lambda\tilde{\alpha})_E(x) = ((\lambda\tilde{\alpha})_D \,\tilde{\cap}\, (\lambda\tilde{\alpha})_E)(x).$$

Assume that $x \notin D \cap E$. Then $(\lambda\tilde{\alpha})_{D\cap E}(x) = \emptyset$. If $x \notin D$, then

$$((\lambda\tilde{\alpha})_D \,\tilde{\cap}\, (\lambda\tilde{\alpha})_E)(x) = (\lambda\tilde{\alpha})_D(x) \cap (\lambda\tilde{\alpha})_E(x) = \emptyset = (\lambda\tilde{\alpha})_{D\cap E}(x).$$

Similarly, if $x \notin E$, then $((\lambda\tilde{\alpha})_D \,\tilde{\cap}\, (\lambda\tilde{\alpha})_E)(x) = (\lambda\tilde{\alpha})_{D\cap E}(x)$. Therefore

$$((\lambda\tilde{\alpha})_D \,\tilde{\cap}\, (\lambda\tilde{\alpha})_E, T) = ((\lambda\tilde{\alpha})_{D\cap E}, S).$$

(3) Let $x \in T$. If $x \in D \cup E$, then $x \in D$ or $x \in E$, which implies that $(\lambda\tilde{\alpha})_D(x) = \lambda$ or $(\lambda\tilde{\alpha})_E(x) = \lambda$. Hence

$$(\lambda\tilde{\alpha})_{D\cup E}(x) = \lambda = (\lambda\tilde{\alpha})_D(x) \cup (\lambda\tilde{\alpha})_E(x) = ((\lambda\tilde{\alpha})_D \,\tilde{\cup}\, (\lambda\tilde{\alpha})_E)(x).$$

Suppose that $x \notin D \cup E$. Then $x \notin D$ and $x \notin E$. It follows that

$$((\lambda\tilde{\alpha})_D \,\tilde{\cup}\, (\lambda\tilde{\alpha})_E)(x) = (\lambda\tilde{\alpha})_D(x) \cup (\lambda\tilde{\alpha})_E(x) = \emptyset = (\lambda\tilde{\alpha})_{D\cup E}(x).$$

Therefore $((\lambda\tilde{\alpha})_D \,\tilde{\cup}\, (\lambda\tilde{\alpha})_E, T) = ((\lambda\tilde{\alpha})_{D\cup E}, T)$.

(4) Let $x \in T$. If $x \in (D]$, then $x \leq b$ for some $b \in D$. Hence

$$\left(\tilde{\bigcup_{a\in D}} a_\lambda\right)(x) = \bigcup_{a\in D} a_\lambda(x) \supseteq b_\lambda(x) = \lambda. \tag{3.5}$$

Note that $a_\lambda(x) \subseteq \lambda$ for any ordered soft point a_λ over U. Thus

$$\left(\tilde{\bigcup_{a\in D}} a_\lambda\right)(x) = \bigcup_{a\in D} a_\lambda(x) \subseteq \lambda. \tag{3.6}$$

Conditions (3.5) and (3.6) induce $(\tilde{\bigcup}_{a\in D} a_\lambda)(x) = \lambda = (\lambda\tilde{\alpha})_{(D]}(x)$. If $x \notin (D]$, then

(i) $(\lambda\tilde{\alpha})_{(D]}(x) = \emptyset$ and
(ii) $x \notin (a]$ for all $a \in D$, and so $a_\lambda(x) = \emptyset$ for all $a \in D$.

It follows that

$$\left(\tilde{\bigcup_{a\in D}} a_\lambda\right)(x) = \bigcup_{a\in D} a_\lambda(x) = \emptyset = (\lambda\tilde{\alpha})_{(D]}(x).$$

Therefore $((\lambda\tilde{\alpha})_{(D]}, T) = (\tilde{\bigcup}_{a\in D} a_\lambda, T)$. □

Theorem 3.17. *If D is a left ideal of T, then $(\lambda\tilde{\alpha})_D$ is an int-soft l-ideal of T.*

Proof. Suppose that D is a left ideal of T. Let $x, y \in T$. If $y \in D$, then $xy \in D$ and so $(\lambda\tilde{\alpha})_D(xy) = \lambda = (\lambda\tilde{\alpha})_D(y)$. If $y \notin D$, then $(\lambda\tilde{\alpha})_D(y) = \emptyset \subseteq (\lambda\tilde{\alpha})_D(xy)$. Assume that $x \leq y$. If $y \in D$, then $x \in D$ and thus $(\lambda\tilde{\alpha})_D(x) = \lambda = (\lambda\tilde{\alpha})_D(y)$. If $y \notin D$, then $(\lambda\tilde{\alpha})_D(y) = \emptyset \subseteq (\lambda\tilde{\alpha})_D(x)$. Therefore $(\lambda\tilde{\alpha})_D$ is an int-soft l-ideal of T. □

Theorem 3.18. *If D is a right ideal of T, then $(\lambda\tilde{\alpha})_D$ is an int-soft r-ideal of T.*

Proof. Similar proof as above. □

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