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Research Article

An Approach to BMBJ-Neutrosophic Hyper-BCK-Ideals of Hyper-BCK-Algebras

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In this article, a new idea of BMBJ-neutrosophic hyper-BCK-algebras is introduced and some of its properties are investigated. Here, BMBJ-neutrosophic hyper-BCK-ideal, BMBJ-neutrosophic weak hyper-BCK-ideal, BMBJ-neutrosophic s-weak hyper-BCK-ideal, and BMBJ-neutrosophic strong hyper-BCK-ideal are presented, and some relevant results and relations are indicated. Characterizations of BMBJ-neutrosophic (weak, s-weak, strong) hyper-BCK-ideal are considered. Conditions for a BMBJ-neutrosophic weak hyper-BCK-ideal to be a BMBJ-neutrosophic s-weak hyper-BCK-ideal are provided. Conditions for an MBJ-neutrosophic set to be a BMBJ-neutrosophic strong hyper-BCK-ideal are given.

1. Introduction

A hypergroup, as a generalization of a group, was introduced by Marty [1] in 1934. Many authors have developed the discussion of hyperstructures (also called multialgebras), such as Corsini [2] and Vougiouklis [3]. We can find well-written books for the introduction to hyperstructures, e.g., Corsini [2], Corsini and Leoreanu [4], Davvaz [5, 6], Davvaz and Cristea [7], and Schweigert [8]. Another topic, which has roused the interest of several mathematicians, is that one of hyper-BCK-algebra (briefly, $\mathbb{H}$-BCK-algebra), introduced by Jun et al. [9]. $\mathbb{H}$-BCK-algebras represent a natural extension of classical BCK-algebras. In a classical BCK-algebra, the composition of two elements is an element, while in a $\mathbb{H}$-BCK-algebra, the composition of two elements is a set.

As an extension of the classical notion of a set, Zadeh [10], in 1965, proposed fuzzy sets (briefly, FSs) as mathematical model of vagueness where elements belong to a given set to some degree that is typically a number that belongs to the unit interval $[0, 1]$. In 1986, this concept has been generalized to intuitionistic fuzzy set (briefly, IFS) theory by adding a nonmembership function by Atanassov [11]. On the other hand, neutrosophy is an almost new branch in pure mathematics which was introduced in 1998 by Smarandache (see [12, 13]). It is an extension of the classical idea of a set and it is related to IFS theory and intuitionistic logic. Neutrosophic sets (briefly, NSs) are sets whose elements have independent degrees of truth and indeterminate and false memberships in the unit interval $[0, 1]$. The natural generalization of NS theory is the approach of MBJ-neutrosophic sets (briefly, MBJ-NSs), introduced by Takallo et al. [14] in 2018, when they generalized the indeterminacy membership function in a NS to an interval-valued membership function. If the interval-valued indeterminacy membership function of an MBJ-NS takes equal lower and upper values, then we go back to the NS. Thus, MBJ-NSs provide a more adequate description of uncertainty than NSs. Different extensions of FSs have been extensively implemented to algebraic structures, decision-making problems, etc. For algebraic structures (especially, BCK/BCI-algebras and semigroups), see [15–21], and for decision-making problems, see [22–24]. In [14], Takallo et al.

introduced the concept of an MBJ-N subalgebra as a generalization of a neutrosophic subalgebra in BCK/BCI-algebras and next Jun and Roh [25] introduced and studied the concept of an MBJ-N ideal. In B-algebras, Manokaran and Prakasam [26] introduced the MBJ-N subalgebra and Khalid et al. [27] defined and studied the MBJ-N T-ideal. As a new idea and based on MBJ-NSs, Bordbar et al. [28] proposed the notion of a BMBJ-N subalgebra and Takallo et al. [29] introduced the notions of a BMBJ-N°-subalgebra and a (closed) BMBJ-N ideal in BCK/BCI-algebras. Borzooei et al. [30] presented a positive implicative BMBJ-N ideal in BCK-algebras.

In an algebraic hyperstructure, Jun and Xin [31] first studied the fuzzy $\mathbb{H}$-BCK-ideal of a $\mathbb{H}$-BCK-algebra and Bakhshi et al. [32] introduced fuzzy (weak, positive) implicative $\mathbb{H}$-BCK-ideals and then the fuzzification of $\mathbb{H}$-BCK-algebra started to grow up. In particular, a link between hyper $\mathbb{H}$-BCK-algebras and IFSs has been established by Borzooei and Jun [33] where they discussed intuitionistic fuzzy $\mathbb{H}$-BCK-ideals of $\mathbb{H}$-BCK-algebras. Later on, other researchers considered this field of study such as Jun [34] who discussed fuzzy $\mathbb{H}$-BCK-ideals of $\mathbb{H}$-BCK-algebras with multivalued membership functions. Also, an article was written by Seo et al. on $\mathbb{H}$-BCK-algebras and multipolar intuitionistic fuzzy $\mathbb{H}$-BCK-ideals (see [35]). As a generalization of fuzzification of hyperalgebraic structures, some researchers started working on fuzzification of hyperalgebraic structures. In fact, a link between NSs and hyperalgebraic structures was recently established and some work was done in this regard, see [36–39].

In our paper, we combine the notion of $\mathbb{H}$-BCK-algebras with MBJ-NSs to define some types of BMBJ-neutrosophic $\mathbb{H}$-BCK-ideals (briefly, BMBJ-N $\mathbb{H}$-BCK-ideals) of $\mathbb{H}$-BCK-algebras and it is organized as follows: after an introduction, Section 2 briefly reviews some preliminary results related to $\mathbb{H}$-BCK-algebras and MBJ-NSs that are used throughout the paper. Section 3 defines the notions of BMBJ-neutrosophic weak hyper-BCK-ideals (briefly, BMBJ-N $\mathbb{WH}$-BCK-ideals), BMBJ-neutrosophic s-weak hyper-BCK-ideals (briefly, BMBJ-N s-$\mathbb{WH}$-BCK-ideals), and BMBJ-neutrosophic strong hyper-BCK-ideals (briefly, BMBJ-N $\mathcal{S}\mathbb{H}$-BCK-ideals) of $\mathbb{H}$-BCK-algebras and presents several results related to the new defined concepts. Also, we discuss BMBJ-N $\mathbb{WH}$-BCK-ideal and BMBJ-N $\mathcal{S}\mathbb{H}$-BCK-ideal in relation to level cut sets. We find conditions for a BMBJ-N $\mathbb{WH}$-BCK-ideal to be a BMBJ-N s-$\mathbb{WH}$-BCK-ideal. We give conditions for an MBJ-NS to be a BMBJ-N $\mathcal{S}\mathbb{H}$-BCK-ideal. Finally, in Section 4, we present the conclusion and future works of the study.

2. Preliminaries

In the current section, we remember some of the basic notions of $\mathbb{H}$-BCK-algebras and MBJ-NSs which will be very helpful in further study of the paper. Let $\widetilde{\mathscr{H}}$ be a $\mathbb{H}$-BCK-algebra in what follows, unless otherwise stated.

Let $\widetilde{\mathscr{H}}$ be a nonempty set and let "$\diamond$" be a mapping

$$\diamond : \widetilde{\mathscr{H}} \times \widetilde{\mathscr{H}} \longrightarrow \mathscr{H}(\widetilde{\mathscr{H}})\backslash\{\phi\}, \tag{1}$$

which is said to be hyperoperation. For any two subsets $\mathscr{K}$ and $\mathscr{F}$, denote by $\mathscr{K} \diamond \mathscr{F}$, the set $\bigcup\{\varrho \diamond \tau | \varrho \in \mathscr{K}, \tau \in \mathscr{F}\}$. W shall use $\varrho \diamond \tau$ instead of $\{\varrho\} \diamond \tau$, $\varrho \diamond \{\tau\}$, or $\{\varrho\} \diamond \{\tau\}$.

By a $\mathbb{H}$-BCK-algebra $\widetilde{\mathscr{H}}$ (see [9]), we mean a set $\widetilde{\mathscr{H}}\ (\neq \phi)$ with a special element 0 and a hyperoperation $\diamond$, for all $\varrho, \tau, \eta \in \widetilde{\mathscr{H}}$, that satisfies the following axioms:

(HI) $(\varrho \diamond \eta) \diamond (\tau \diamond \eta) = \varrho \diamond \tau$,

(HII) $(\varrho \diamond \tau) \diamond \eta = (\varrho \diamond \eta) \diamond \tau$,

(HIII) $\varrho \diamond \widetilde{\mathscr{H}} \ll \{\varrho\}$,

(HIV) $\varrho \ll \tau$ and $\tau \ll \varrho$ imply $\varrho = \tau$,

For all $\varrho, \tau, \eta \in \widetilde{\mathscr{H}}$, where $\varrho \ll \tau$ is defined by $0 \in \varrho \diamond \tau$ and $\forall \mathscr{K}, \mathscr{F} \subseteq \widetilde{\mathscr{H}}$, $\mathscr{K} \ll \mathscr{F}$ is defined by $\forall r \in \mathscr{K}$, $\exists t \in \mathscr{F}$ such that $r \ll t$.

In a $\mathbb{H}$-BCK-algebra $\widetilde{\mathscr{H}}$, the axiom (HIII) is equivalent to (HV), where

(HV) $\varrho \diamond \tau \ll \{\varrho\}$ for all $\varrho, \tau \in \widetilde{\mathscr{H}}$.

Proposition 1 (see [9]). *Every $\mathbb{H}$-BCK-algebra $\widetilde{\mathscr{H}}$ satisfies the following conditions, for all $\varrho, \tau, \eta \in \widetilde{\mathscr{H}}$ and for any nonempty subsets $\mathscr{K}, \mathscr{F}, \mathscr{G}$ of $\widetilde{\mathscr{H}}$,*

(1) $\varrho \diamond 0 \ll \{\varrho\}$, $0 \diamond \varrho = \{0\}$, $0 \diamond 0 = \{0\}$,

(2) $0 \ll \varrho$, $\varrho \ll \varrho$, $\varrho \in \varrho \diamond 0$,

(3) $\varrho \diamond 0 \ll \{\tau\} \Rightarrow \varrho \ll \tau$,

(4) $\tau \ll \eta \Rightarrow \varrho \diamond \eta \ll \varrho \diamond \tau$,

(5) $\varrho \diamond \tau = \{0\} \Rightarrow \varrho \diamond \eta \ll \tau \diamond \eta$, $(\varrho \diamond \eta) \diamond (\tau \diamond \eta) = \{0\}$,

(6) $\mathscr{K} \subseteq \mathscr{F} \Rightarrow \mathscr{K} \ll \mathscr{F}$,

(7) $\mathscr{K} \ll \{0\} \Rightarrow \mathscr{K} = \{0\}$,

(8) $\mathscr{K} \ll \mathscr{K}$, $\mathscr{K} \diamond \mathscr{F} \ll \mathscr{K}$, $(\mathscr{K} \diamond \mathscr{F}) \diamond \mathscr{G} = (\mathscr{K} \diamond \mathscr{G}) \diamond \mathscr{F}$,

(9) $\mathscr{K} \diamond \{0\} = \{0\} \Rightarrow \mathscr{K} = \{0\}$.

Definition 1. Let $(\widetilde{\mathscr{H}}, \diamond)$ be a $\mathbb{H}$-BCK-algebra. A subset $\mathscr{K}$ of $\widetilde{\mathscr{H}}$ is called as follows:

A hyper-BCK-ideal (briefly, $\mathbb{H}$-BCK-ideal) of $\widetilde{\mathscr{H}}$ (see [9]) if

(1) $0 \in \mathscr{K}$,
(2) $\varrho \diamond \tau \ll \mathscr{K}$, $\tau \in \mathscr{K} \Rightarrow \varrho \in \mathscr{K}, \forall \varrho, \tau \widetilde{\mathscr{H}}$.

A weak hyper-BCK-ideal (briefly, $\mathbb{WH}$-BCK-ideal) of $\widetilde{\mathscr{H}}$ (see [9]), if it satisfies (1) and

(3) $\varrho \diamond \tau \subseteq \mathscr{K}$, $\tau \in \mathscr{K} \Rightarrow \varrho \in \mathscr{K}, \forall \varrho, \tau \in \widetilde{\mathscr{H}}$,

A strong hyper-BCK-ideal (briefly, $\mathcal{S}\mathbb{H}$-BCK-ideal) of $\widetilde{\mathscr{H}}$ (see [40]), if it satisfies (1) and

(4) $\varrho \diamond \tau \cap \mathscr{K} \neq \phi$, $\tau \in \mathscr{K} \Rightarrow \varrho \in \mathscr{K}, \forall \varrho, \tau \in \widetilde{\mathscr{H}}$.

By an interval number $\widetilde{\varpi}$, we mean an interval $\widetilde{\varpi} = [\varpi^-, \varpi^+]$, where $0 \leq \varpi^- \leq \varpi^+ \leq 1$. The set of all closed interval numbers I is denoted by $[I]$. The interval $[\varpi, \varpi]$ is identified with the number ϖ.

For two interval numbers $\widetilde{\varpi}_1 = [\varpi_1^-, \varpi_1^+]$ and $\widetilde{\varpi}_2 = [\varpi_2^-, \varpi_2^+]$, we define

$$\begin{aligned} r\max\{\widetilde{\varpi}_1, \widetilde{\varpi}_2\} &= [\max\{\varpi_1^-, \varpi_2^-\}, \max\{\varpi_1^+, \varpi_2^+\}],\\ r\min\{\widetilde{\varpi}_1, \widetilde{\varpi}_2\} &= [\min\{\varpi_1^-, \varpi_2^-\}, \min\{\varpi_1^+, \varpi_2^+\}], \end{aligned} \tag{2}$$

Furthermore, we have

(1) $\widetilde{\varpi}_1 \pm \widetilde{\varpi}_2 \Leftrightarrow \varpi_1^- \geq \varpi_2^-,\ \varpi_1^+ \geq \varpi_2^+$,
(2) $\widetilde{\varpi}_1 \prec \widetilde{\varpi}_2 \Leftrightarrow \varpi_1^- \leq \varpi_2^-,\ \varpi_1^+ \leq \varpi_2^+$,
(3) $\widetilde{\varpi}_1 = \widetilde{\varpi}_2 \Leftrightarrow \varpi_1^- = \varpi_2^-,\ \varpi_1^+ = \varpi_2^+$.

Let $\widetilde{\mathscr{H}}$ be a nonempty set. A function $\widetilde{D}$: $\widetilde{\mathscr{H}} \longrightarrow [I]$ is called an interval-valued fuzzy set (briefly, IVFS) over a universe $\widetilde{\mathscr{H}}$.

Let $[I]^H$ stand for the set of all IVFSs $\widetilde{\mathscr{H}}$. For any $\widetilde{D} \in [I]^H$ and $\varrho \in \widetilde{\mathscr{H}}$, $\widetilde{D} = [D^-(\varrho), D^+(\varrho)]$ is called the degree of membership of an element ϱ to $\widetilde{D}$, where $D^-(\varrho)$: $\widetilde{\mathscr{H}} \longrightarrow I$ and $D^+(\varrho)$: $\widetilde{\mathscr{H}} \longrightarrow I$ are FSs over a universe $\widetilde{\mathscr{H}}$ which are called a lower FS and an upper FS over $\widetilde{\mathscr{H}}$, respectively. For simplicity, we denote $\widetilde{D} = [D^-, D^+]$.

Let $\widetilde{\mathscr{H}}$ be a nonempty set. An NS over a universe $\widetilde{\mathscr{H}}$ (see [12]) is a structure of the form:

$$\mathscr{D} = \left\{\langle \varrho; \mathscr{D}_T(\varrho), \mathscr{D}_I(\varrho), \mathscr{D}_F(\varrho)\rangle | \varrho \in \widetilde{\mathscr{H}}\right\}, \tag{3}$$

where $\mathscr{D}_T$, $\mathscr{D}_I$, and $\mathscr{D}_I$ are FSs over a universe $\widetilde{\mathscr{H}}$, which are called a truth and an indeterminate and false membership functions, respectively.

For the sake of simplicity, we shall use the symbol $\mathscr{D} = (\mathscr{D}_T, \mathscr{D}_I, \mathscr{D}_F)$ for the NS

$$\mathscr{D} = \left\{\langle \varrho; \mathscr{D}_T(\varrho), \mathscr{D}_I(\varrho), \mathscr{D}_F(\varrho)\rangle | \varrho \in \widetilde{\mathscr{H}}\right\}. \tag{4}$$

In [14], Takallo et al. introduced the idea of an MBJ-NS as follows.

Definition 2. Let $\widetilde{\mathscr{H}}$ be a nonempty set. By an MBJ-NS over a universe $\widetilde{\mathscr{H}}$, we mean a structure of the following form:

$$\mathscr{W} = \left\{\langle \varrho; \mathscr{M}_{\mathscr{W}}(\varrho), \widetilde{\mathscr{B}}_{\mathscr{W}}(\varrho), \mathscr{J}_{\mathscr{W}}(\varrho)\rangle | \varrho \in \widetilde{\mathscr{H}}\right\}, \tag{5}$$

where $\mathscr{M}_{\mathscr{W}}$ and $\mathscr{J}_{\mathscr{W}}$ are FSs over a universe $\widetilde{\mathscr{H}}$, which are called a truth and a false membership functions, respectively, and $\widetilde{\mathscr{B}}_{\mathscr{W}}$ is an interval-valued fuzzy set over a universe $\mathscr{H}$ which is called an indeterminate interval-valued membership function.

For the sake of simplicity, we shall use the symbol $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ for the MBJ-NS,

$$\mathscr{W} = \left\{\langle \varrho; \mathscr{M}_{\mathscr{W}}(\varrho), \widetilde{\mathscr{B}}_{\mathscr{W}}(\varrho), \mathscr{J}_{\mathscr{W}}(\varrho)\rangle | \varrho \in \widetilde{\mathscr{H}}\right\}. \tag{6}$$

Given an MBJ-NS $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ over a universe $\widetilde{\mathscr{H}}$, we consider the following sets:

$$\begin{aligned} U(\mathscr{M}_{\mathscr{W}}, \alpha) &= \{\varrho \in H | \mathscr{M}_{\mathscr{W}}(\varrho) \geq \alpha\},\\ L(\mathscr{B}^-_{\mathscr{W}}, \delta^-) &= \{\varrho \in H | \mathscr{B}^-_{\mathscr{W}}(\varrho) \leq \delta^-\},\\ U(\mathscr{B}^+_{\mathscr{W}}, \delta^+) &= \{\varrho \in H | \mathscr{B}^+_{\mathscr{W}}(\varrho) \geq \delta^+\},\\ L(\mathscr{J}_{\mathscr{W}}, \gamma) &= \{\varrho \in H | \mathscr{J}_{\mathscr{W}}(\varrho) \leq \gamma\}, \end{aligned} \tag{7}$$

where $\alpha, \gamma, \delta^-, \delta^- \in [0, 1]$.

3. BMBJ-Neutrosophic Hyper-BCK-Ideals

Definition 3. An MBJ-NS $\mathscr{W}$ on $\widetilde{\mathscr{H}}$ is called a BMBJ-N $\mathbb{H}$-BCK-ideal of $\widetilde{\mathscr{H}}$, if it satisfies

(1) $(\forall \varrho, \tau \in \widetilde{\mathscr{H}}) \left(\varrho \ll \tau \Rightarrow \begin{pmatrix} \mathscr{M}_{\mathscr{W}}(\varrho) \geq \mathscr{M}_{\mathscr{W}}(\tau) \\ \mathscr{B}^-_{\mathscr{W}}(\varrho) \leq \mathscr{B}^-_{\mathscr{W}}(\tau) \\ \mathscr{B}^+_{\mathscr{W}}(\varrho) \geq \mathscr{B}^+_{\mathscr{W}}(\tau) \\ \mathscr{J}_{\mathscr{W}}(\varrho) \leq \mathscr{J}_{\mathscr{W}}(\tau) \end{pmatrix} \right)$,

(2) $(\forall \varrho, \tau \in \widetilde{\mathscr{H}})$ $(\mathscr{M}_{\mathscr{W}}(\varrho) \geq \min\{\inf\{\mathscr{M}_{\mathscr{W}}(z) | z \in \varrho \diamond \tau\}, \mathscr{M}_{\mathscr{W}}(\tau)\} \mathscr{B}^-_{\mathscr{W}}(\varrho) \leq \max\{\sup\{\mathscr{B}^-_{\mathscr{W}}(z) | z \in \varrho \diamond \tau\}, \mathscr{B}^-_{\mathscr{W}}(\tau)\} \mathscr{B}^+_{\mathscr{W}}(\varrho) \geq \min\{\inf\{\mathscr{B}^+_{\mathscr{W}}(z) | z \in \varrho \diamond \tau\}, \mathscr{B}^+_{\mathscr{W}}(\tau)\} \mathscr{J}_{\mathscr{W}}(\varrho) \leq \max\{\sup\{\mathscr{J}_{\mathscr{W}}(z) | z \in \varrho \diamond \tau\}, \mathscr{J}_{\mathscr{W}}(\tau)\})$.

Example 1. Let $\widetilde{\mathscr{H}} = \{0, \varrho, \tau\}$ be a set with the hyperoperation " $\diamond$ ", which is given by Table 1.

Then, $\widetilde{\mathscr{H}}$ is a $\mathbb{H}$-BCK-algebra. Let $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ be an MBJ-NS over $\widetilde{\mathscr{H}}$ given by Table 2.

It is routine to check that $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ is a BMBJ-N $\mathbb{H}$-BCK-ideal of $\widetilde{\mathscr{H}}$.

Proposition 2. *Let $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ be a BMBJ-N $\mathbb{H}$-BCK-ideal of $\widetilde{\mathscr{H}}$. Then,*

(i) $(\forall \varrho \in \widetilde{\mathscr{H}}) \begin{pmatrix} \mathscr{M}_{\mathscr{W}}(0) \geq \mathscr{M}_{\mathscr{W}}(\varrho) \\ \mathscr{B}^-_{\mathscr{W}}(0) \leq \mathscr{B}^-_{\mathscr{W}}(\varrho) \\ \mathscr{B}^+_{\mathscr{W}}(0) \geq \mathscr{B}^+_{\mathscr{W}}(\varrho) \\ \mathscr{J}_{\mathscr{W}}(0) \leq \mathscr{J}_{\mathscr{W}}(\varrho) \end{pmatrix}$.

(ii) If $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ satisfies

$$(\forall \mathscr{K} \subseteq \widetilde{\mathscr{H}})(\exists \varrho_\circ, \tau_1, \tau_2, {}_{\eta^\circ} \in \mathscr{K}) \begin{pmatrix} \mathscr{M}_{\mathscr{W}}(\varrho_\circ) = \inf\{\mathscr{M}_{\mathscr{W}}(\varrho) | \varrho \in \mathscr{K}\} \\ \mathscr{B}^-_{\mathscr{W}}(\tau_1) = \sup\{\mathscr{B}^-_{\mathscr{W}}(\tau) | \tau \in \mathscr{K}\} \\ \mathscr{B}^+_{\mathscr{W}}(\tau_2) = \inf\{\mathscr{B}^+_{\mathscr{W}}(\tau) | \tau \in \mathscr{K}\} \\ \mathscr{J}_{\mathscr{W}}({}_{\eta^\circ}) = \sup\{\mathscr{J}_{\mathscr{W}}(\eta) | \eta \in \mathscr{K}\} \end{pmatrix}, \tag{8}$$

Table 1: Tabular representation of the hyperoperation "$\diamond$."

$\diamond$	0	ϱ	τ
0	$\{0\}$	$\{0\}$	$\{0\}$
ϱ	$\{\varrho\}$	$\{0,\varrho\}$	$\{0,\varrho\}$
τ	$\{\tau\}$	$\{\varrho,\tau\}$	$\{0,\varrho,\tau\}$

Table 2: Tabular representation of $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$.

$\mathscr{W}$	$\mathscr{M}_{\mathscr{W}}$	$\widetilde{\mathscr{B}}_{\mathscr{W}}$	$\mathscr{J}_{\mathscr{W}}$
0	1/5	[(1/4), 0.71]	0.33
ϱ	1/7	[(1/3), 0.51]	0.43
τ	1/9	[(1/3), 0.21]	0.53

then

$$(\forall \varrho,\tau \in \widetilde{\mathscr{H}})(\exists u, v_1, v_2, w \in \varrho \diamond \tau)\begin{pmatrix} \mathscr{M}_{\mathscr{W}}(\varrho) \geq \min\{\mathscr{M}_{\mathscr{W}}(u), \mathscr{M}_{\mathscr{W}}(\tau)\} \\ \mathscr{B}^-_{\mathscr{W}}(\varrho) \leq \max\{\mathscr{B}^-_{\mathscr{W}}(v_1), \mathscr{B}^-_{\mathscr{W}}(\tau)\} \\ \mathscr{B}^+_{\mathscr{W}}(\varrho) \geq \min\{\mathscr{B}^+_{\mathscr{W}}(v_2), \mathscr{B}^+_{\mathscr{W}}(\tau)\} \\ \mathscr{J}_{\mathscr{W}}(\varrho) \leq \max\{\mathscr{J}_{\mathscr{W}}(w), \mathscr{M}_{\mathscr{W}}(\tau)\} \end{pmatrix}. \tag{9}$$

Proof. The proof is obvious and is omitted. □

Corollary 1. *In a finite-BCK-algebra, every BMBJ-N $\mathbb{H}$-BCK-ideal $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ over $\widetilde{\mathscr{H}}$ satisfies condition (9).*

Lemma 1 (see [41]). *Let $\mathscr{K}$ be a subset of a $\mathbb{H}$-BCK-algebra $\widetilde{\mathscr{H}}$. If $\mathscr{I}$ is a $\mathbb{H}$-BCK-ideal of $\widetilde{\mathscr{H}}$ such that $\mathscr{K} \ll \mathscr{I}$, then $\mathscr{K}$ is contained in $\mathscr{I}$.*

Theorem 1. *An MBJ-NS $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ over $\widetilde{\mathscr{H}}$ is a BMBJ-N $\mathbb{H}$-BCK-ideal of $\widetilde{\mathscr{H}}$ if and only if the nonempty sets $U(\mathscr{M}_{\mathscr{W}}, \alpha)$, $L(\mathscr{B}^-_{\mathscr{W}}, \delta^-)$, $U(\mathscr{B}^+_{\mathscr{W}}, \delta^+)$, and $L(\mathscr{J}_{\mathscr{W}}, \gamma)$ are $\mathbb{H}$-BCK-ideals of $\mathscr{H}$ for any $\alpha, \delta^-, \delta^+, \gamma \in [0,1]$.*

Proof. Assume that $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ is a BMBJ-N $\mathbb{H}$-BCK-ideal of $\widetilde{\mathscr{H}}$. Let $\alpha, \delta^-, \delta^+, \gamma \in [0,1]$ be such that $U(\mathscr{M}_{\mathscr{W}}, \alpha)$, $L(\mathscr{B}^-_{\mathscr{W}}, \delta^-)$, $U(\mathscr{B}^+_{\mathscr{W}}, \delta^+)$, and $L(\mathscr{J}_{\mathscr{W}}, \gamma)$ are nonempty sets. Clearly, $0 \in U(\mathscr{M}_{\mathscr{W}}, \alpha)$, $0 \in L(\mathscr{B}^-_{\mathscr{W}}, \delta^-)$, $0 \in U(\mathscr{B}^+_{\mathscr{W}}, \delta^+)$, and $0 \in L(\mathscr{J}_{\mathscr{W}}, \gamma)$ by Proposition 2(i). Let $\varrho, \tau, u_1, v_1, u_2, v_2, a, b \in \widetilde{\mathscr{H}}$ be such that $\varrho \diamond \tau \in U(\mathscr{M}_{\mathscr{W}}, \alpha)$, $\tau \in U(\mathscr{M}_{\mathscr{W}}, \alpha)$, $u_1 \diamond v_1 \in L(\mathscr{B}^-_{\mathscr{W}}, \delta^-)$, $v_1 \in L(\mathscr{B}^-_{\mathscr{W}}, \delta^-)$, $u_2 \diamond v_2 \in U(\mathscr{B}^+_{\mathscr{W}}, \delta^+)$, $v_2 \in U(\mathscr{B}^+_{\mathscr{W}}, \delta^+)$, $a \diamond b \in L(\mathscr{J}_{\mathscr{W}}, \gamma)$, and $b \in L\ (\mathscr{J}_{\mathscr{W}}, \gamma)$. Then, $\varrho \diamond \tau \ll U(\mathscr{M}_{\mathscr{W}}, \alpha)$, $\tau \in U(\mathscr{M}_{\mathscr{W}}, \alpha)$, $u_1 \diamond v_1 \ll L(\mathscr{B}^-_{\mathscr{W}}, \delta^-)$, $v_1 \in L(\mathscr{B}^-_{\mathscr{W}}, \delta^-)$, $u_2 \diamond v_2 \ll U(\mathscr{B}^+_{\mathscr{W}}, \delta^+)$, $v_2 \in U(\mathscr{B}^+_{\mathscr{W}}, \delta^+)$, $a \diamond b \ll L(\mathscr{J}_{\mathscr{W}}, \gamma)$, and $b \in L(\mathscr{J}_{\mathscr{W}}, \gamma)$. It follows that

$$\begin{aligned} &(\forall x \in \varrho \diamond \tau)(\exists_{x^\circ} \in U(\mathscr{M}_{\mathscr{W}}, \alpha) \text{ such that } x \ll x^\circ) \text{ and so } \mathscr{M}_{\mathscr{W}}(x) \geq \mathscr{M}_{\mathscr{W}}(x^\circ), \\ &(\forall y_1 \in u_1 \diamond v_1)(\exists y_{1^\circ} \in L(\mathscr{B}^-_{\mathscr{W}}, \delta^-) \text{ such that } y_1 \ll y_{1^\circ}) \text{ and so } \mathscr{B}^-_{\mathscr{W}}(y_1) \leq \mathscr{B}^-_{\mathscr{W}}(y_{1^\circ}), \\ &(\forall y_2 \in u_2 \diamond v_2)(\exists y_{2^\circ} \in U(\mathscr{B}^+_{\mathscr{W}}, \delta^+) \text{ such that } y_2 \ll y_{2^\circ}) \text{ and so } \mathscr{B}^+_{\mathscr{W}}(y_2) \geq \mathscr{B}^+_{\mathscr{W}}(y_{2^\circ}), \\ &(\forall z \in a \diamond b)(\exists_{z^\circ} \in L(\mathscr{J}_{\mathscr{W}}, \gamma) \text{ such that } z \ll z^\circ) \text{ and so } \mathscr{J}_{\mathscr{W}}(z) \leq \mathscr{J}_{\mathscr{W}}(z^\circ), \end{aligned} \tag{10}$$

which imply that $\mathscr{M}_{\mathscr{W}}(x) \geq \alpha$, $\mathscr{B}^-_{\mathscr{W}}(y_1) \leq \delta^-$, $\mathscr{B}^+_{\mathscr{W}}(y_2) \geq \delta^+$, and $\mathscr{J}_{\mathscr{W}}(z) \leq \gamma$ for all $x \in \varrho \diamond \tau$, $y_1 \in u_1 \diamond v_1$, $y_2 \in u_2 \diamond v_2$, and $z \in a \diamond b$. Hence, $\inf\{\mathscr{M}_{\mathscr{W}}(x) | x \in \varrho \diamond \tau\} \geq \alpha$, $\sup\{\mathscr{B}^-_{\mathscr{W}}(y_1) \ | y_1 \in u_1 \diamond v_1\} \leq \delta^-$, $\inf\{\mathscr{B}^+_{\mathscr{W}}(y_2) | y_2 \in u_2 \diamond v_2\} \geq \delta^+$, and $\sup\{\mathscr{J}_{\mathscr{W}}\ (z) | z \in a \diamond b\} \leq \gamma$ and so

$$\begin{aligned} \mathscr{M}_{\mathscr{W}}(\varrho) &\geq \min\{\inf\{\mathscr{M}_{\mathscr{W}}(x) | x \in \varrho \diamond \tau\}, \mathscr{M}_{\mathscr{W}}(\tau)\} \geq \min\{\alpha, \alpha\} = \alpha, \\ \mathscr{B}^-_{\mathscr{W}}(u_1) &\leq \max\{\sup\{\mathscr{B}^-_{\mathscr{W}}(y_1) | y_1 \in u_1 \diamond v_1\}, \mathscr{B}^-_{\mathscr{W}}(v_1)\} \leq \max\{\delta^-, \delta^-\} = \delta^-, \\ \mathscr{B}^+_{\mathscr{W}}(u_2) &\geq \min\{\inf\{\mathscr{B}^+_{\mathscr{W}}(y_2) | y_2 \in u_2 \diamond v_2\}, \mathscr{B}^+_{\mathscr{W}}(v_2)\} \geq \min\{\delta^+, \delta^+\} = \delta^+, \\ \mathscr{J}_{\mathscr{W}}(a) &\leq \max\{\sup\{\mathscr{J}_{\mathscr{W}}(z) | z \in a \diamond b\}, \mathscr{J}_{\mathscr{W}}(b)\} \leq \max\{\gamma, \gamma\} = \gamma. \end{aligned} \tag{11}$$

Thus, $\varrho \in U(\mathscr{M}_{\mathscr{W}},\alpha)$, $u_1 \in L(\mathscr{B}^-_{\mathscr{W}},\delta^-)$, $u_2 \in U(\mathscr{B}^+_{\mathscr{W}},\delta^+)$, and $a \in L(\mathscr{J}_{\mathscr{W}},\gamma)$, and hence, $U(\mathscr{M}_{\mathscr{W}},\alpha)$, $L(\mathscr{B}^-_{\mathscr{W}},\delta^-)$, $U(\mathscr{B}^+_{\mathscr{W}},\delta^+)$, and $L(\mathscr{J}_{\mathscr{W}},\gamma)$ are $\mathbb{H}$-BCK-ideals of $\widetilde{\mathscr{H}}$.

Conversely, suppose the nonempty sets $U(\mathscr{M}_{\mathscr{W}},\alpha)$, $L(\mathscr{B}^-_{\mathscr{W}},\delta^-)$, $U(\mathscr{B}^+_{\mathscr{W}},\delta^+)$, and $L(\mathscr{J}_{\mathscr{W}},\gamma)$ are $\mathbb{H}$-BCK-ideals of $\widetilde{\mathscr{H}}$ for all $\alpha,\delta^-,\delta^+,\gamma \in [0,1]$. Let $\varrho,\tau,u_1,v_1,u_2,v_2,a,b \in \widetilde{\mathscr{H}}$ be such that $\varrho \ll \tau$, $u_1 \ll v_1$, $u_2 \ll v_2$, $a \ll b$, $\mathscr{M}_{\mathscr{W}}(\tau)=\alpha$, $\mathscr{B}^-_{\mathscr{W}}(v_1)=\delta^-$, $\mathscr{B}^+_{\mathscr{W}}(v_2)=\delta^+$, and $\mathscr{J}_{\mathscr{W}}(b)=\gamma$. Then, $\tau \in U(\mathscr{M}_{\mathscr{W}},\alpha)$, $v_1 \in L(\mathscr{B}^-_{\mathscr{W}},\delta^-)$, $v_2 \in U(\mathscr{B}^+_{\mathscr{W}},\delta^+)$, and $b \in L(\mathscr{J}_{\mathscr{W}},\gamma)$, and so $\{\varrho\} \ll U(\mathscr{M}_{\mathscr{W}},\alpha)$, $\{u_1\} \ll L(\mathscr{B}^-_{\mathscr{W}},\delta^-)$, $\{u_2\} \ll U(\mathscr{B}^+_{\mathscr{W}},\delta^+)$, and $\{a\} \ll L(\mathscr{J}_{\mathscr{W}},\gamma)$. From Lemma 1, we have $\{\varrho\} \subseteq U(\mathscr{M}_{\mathscr{W}},\alpha)$, $\{u_1\} \subseteq L(\mathscr{B}^-_{\mathscr{W}},\delta^-)$, $\{u_2\} \subseteq U(\mathscr{B}^+_{\mathscr{W}},\delta^+)$, and $\{a\} \subseteq L(\mathscr{J}_{\mathscr{W}},\gamma)$, i.e., $\varrho \in U(\mathscr{M}_{\mathscr{W}},\alpha)$, $u_1 \in L(\mathscr{B}^-_{\mathscr{W}},\delta^-)$, $u_2 \in U(\mathscr{B}^+_{\mathscr{W}},\delta^+)$, and $a \in L(\mathscr{J}_{\mathscr{W}},\gamma)$. Hence, $\mathscr{M}_{\mathscr{W}}(\varrho) \ge \alpha = \mathscr{M}_{\mathscr{W}}(\tau)$, $\mathscr{B}^-_{\mathscr{W}}(u_1) \le \delta^- = \mathscr{B}^-_{\mathscr{W}}(v_1)$, $\mathscr{B}^+_{\mathscr{W}}(u_2) \ge \delta^+ = \mathscr{B}^+_{\mathscr{W}}(v_2)$, and $\mathscr{J}_{\mathscr{W}}(a) \le \gamma = \mathscr{J}_{\mathscr{W}}(b)$. For any $\varrho,\tau,u,v,a,b \in \mathscr{H}$, let

$$\begin{aligned}
\alpha &= \min\{\inf\{\mathscr{M}_{\mathscr{W}}(t_1) | t_1 \in \varrho \diamond \tau\}, \mathscr{M}_{\mathscr{W}}(\tau)\},\\
\delta^- &= \max\{\sup\{\mathscr{B}^-_{\mathscr{W}}(t_{2^\circ}) | t_{2^\circ} \in u_1 \diamond v_1\}, \mathscr{B}^-_{\mathscr{W}}(v_1)\},\\
\delta^+ &= \min\{\inf\{\mathscr{B}^+_{\mathscr{W}}(t_{3^\circ}) | t_{3^\circ} \in u_2 \diamond v_2\}, \mathscr{B}^+_{\mathscr{W}}(v_2)\},\\
\gamma &= \max\{\sup\{\mathscr{J}_{\mathscr{W}}(t_3) | t_3 \in a \diamond b\}, \mathscr{J}_{\mathscr{W}}(b)\}.
\end{aligned} \tag{12}$$

Then, $\tau \in U(\mathscr{M}_{\mathscr{W}},\alpha)$, $v_1 \in L(\mathscr{B}^-_{\mathscr{W}},\delta^-)$, $v_2 \in U(\mathscr{B}^+_{\mathscr{W}},\delta^+)$, $b \in L(\mathscr{J}_{\mathscr{W}},\gamma)$, and

$$\begin{aligned}
\mathscr{M}_{\mathscr{W}}(t_4) &\ge \inf\{\mathscr{M}_{\mathscr{W}}(t_1) | t_1 \in \varrho \diamond \tau\}\\
&\ge \min\{\inf\{\mathscr{M}_{\mathscr{W}}(t_1) | t_1 \in \varrho \diamond \tau\}, \mathscr{M}_{\mathscr{W}}(\tau)\}\\
&= \alpha,\\
\mathscr{B}^-_{\mathscr{W}}(t_{5^\circ}) &\le \sup\{\mathscr{B}^-_{\mathscr{W}}(t_{2^\circ}) | t_{2^\circ} \in u_1 \diamond v_1\}\\
&\le \max\{\sup\{\mathscr{B}^-_{\mathscr{W}}(t_{2^\circ}) | t_{2^\circ} \in u_1 \diamond v_1\}, \mathscr{B}^-_{\mathscr{W}}(v_1)\}\\
&= \delta^-,\\
\mathscr{B}^+_{\mathscr{W}}(t_{6^\circ}) &\ge \inf\{\mathscr{B}^+_{\mathscr{W}}(t_{3^\circ}) | t_{3^\circ} \in u_2 \diamond v_2\}\\
&\ge \min\{\inf\{\mathscr{B}^+_{\mathscr{W}}(t_{3^\circ}) | t_{3^\circ} \in u_2 \diamond v_2\}, \mathscr{B}^+_{\mathscr{W}}(v_2)\}\\
&= \delta^+,\\
\mathscr{J}_{\mathscr{W}}(t_6) &\le \sup\{\mathscr{J}_{\mathscr{W}}(t_3) | t_3 \in a \diamond b\}\\
&\le \max\{\sup\{\mathscr{J}_{\mathscr{W}}(t_3) | t_3 \in a \diamond b\}, \mathscr{J}_{\mathscr{W}}(b)\}\\
&= \gamma,
\end{aligned} \tag{13}$$

for all $t_4 \in \varrho \diamond \tau$, $t_{5^\circ} \in u_1 \diamond v_1$, $t_{6^\circ} \in u_2 \diamond v_2$, and $t_6 \in a \diamond b$, i.e, $t_4 \in U(\mathscr{M}_{\mathscr{W}},\alpha)$, $t_{5^\circ} \in L(\mathscr{B}^-_{\mathscr{W}},\delta^-)$, $t_{6^\circ} \in U(\mathscr{B}^+_{\mathscr{W}},\delta^+)$, and $t_6 \in L(\mathscr{J}_{\mathscr{W}},\gamma)$. Thus, $\varrho \diamond \tau \subseteq U(\mathscr{M}_{\mathscr{W}},\alpha)$, $u_1 \diamond v_1 \subseteq L(\mathscr{B}^-_{\mathscr{W}},\delta^-)$, $u_2 \diamond v_2 \subseteq U(\mathscr{B}^+_{\mathscr{W}},\delta^+)$, and $a \diamond b \subseteq L(\mathscr{J}_{\mathscr{W}},\gamma)$, which imply from Proposition 1(6) that $\varrho \diamond \tau \ll U(\mathscr{M}_{\mathscr{W}},\alpha)$, $u_1 \diamond v_1 \ll L(\mathscr{B}^-_{\mathscr{W}},\delta^-)$, $u_2 \diamond v_2 \ll U(\mathscr{B}^+_{\mathscr{W}},\delta^+)$, and $a \diamond b \ll L(\mathscr{J}_{\mathscr{W}},\gamma)$. Since $U(\mathscr{M}_{\mathscr{W}},\alpha)$, $L(\mathscr{B}^-_{\mathscr{W}},\delta^-)$, $U(\mathscr{B}^+_{\mathscr{W}},\delta^+)$, and $L(\mathscr{J}_{\mathscr{W}},\gamma)$ are $\mathbb{H}$-BCK-ideals of $\widetilde{\mathscr{H}}$, we have $\varrho \in U(\mathscr{M}_{\mathscr{W}},\alpha)$, $u_1 \in L(\mathscr{B}^-_{\mathscr{W}},\delta^-)$, $u_2 \in U(\mathscr{B}^+_{\mathscr{W}},\delta^+)$, and $a \in L(\mathscr{J}_{\mathscr{W}},\gamma)$, which imply that

$$\begin{aligned}
\mathscr{M}_{\mathscr{W}}(\varrho) \ge \alpha &= \min\{\inf\{\mathscr{M}_{\mathscr{W}}(t_1) | t_1 \in \varrho \diamond \tau\}, \mathscr{M}_{\mathscr{W}}(\tau)\},\\
\mathscr{B}^-_{\mathscr{W}}(u_1) \le \delta^- &= \max\{\sup\{\mathscr{B}^-_{\mathscr{W}}(t_{2^\circ}) | t_{2^\circ} \in u_1 \diamond v_1\}, \mathscr{B}^-_{\mathscr{W}}(v_1)\},\\
\mathscr{B}^+_{\mathscr{W}}(u_2) \ge \delta^+ &= \min\{\inf\{\mathscr{B}^+_{\mathscr{W}}(t_{3^\circ}) | t_{3^\circ} \in u_2 \diamond v_2\}, \mathscr{B}^+_{\mathscr{W}}(v_2)\},\\
\mathscr{J}_{\mathscr{W}}(a) \le \gamma &= \max\{\sup\{\mathscr{J}_{\mathscr{W}}(t_3) | t_3 \in a \diamond b\}, \mathscr{J}_{\mathscr{W}}(b)\}.
\end{aligned} \tag{14}$$

Therefore, $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ is a BMBJ-N $\mathbb{H}$-BCK-ideal of $\widetilde{\mathscr{H}}$. □

Definition 4. An MBJ-NS $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \mathscr{B}^-_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ over $\widetilde{\mathscr{H}}$ is called as follows:

(1) A BMBJ-N $\mathbb{WH}$-BCK-ideal of $\widetilde{\mathscr{H}}$ if it satisfies Proposition 2(i) and Definition 3(2).

(2) A BMBJ-N s-$\mathbb{WH}$-BCK-ideal of $\widetilde{\mathscr{H}}$ if it satisfies Proposition 2(i) and (2).

Theorem 2. *Every BMBJ-N $\mathbb{H}$-BCK-ideal is a BMBJ-N $\mathbb{WH}$-BCK-ideal.*

Proof. Straightforward. □

The converse of Theorem 2 is not true in general, as seen in the following example.For the converse of Theorem 3, it is not easy to find an example of a BMBJ-N $\mathbb{WH}$-BCK-ideal which is not a BMBJ-N s-$\mathbb{WH}$-BCK-ideal. So, we give the following theorem.

Example 2. Let $\widetilde{\mathscr{H}} = \{0,\varrho,\tau\}$ be a $\mathbb{H}$-BCK-algebra as in Example 1. Let $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ be an MBJ-NS over $\widetilde{\mathscr{H}}$ given by Table 3.

Then, $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ is a BMBJ-N $\mathbb{WH}$-BCK-ideal of $\widetilde{\mathscr{H}}$. Note that $\varrho \ll \tau$,

$$\begin{aligned}
\mathscr{M}_{\mathscr{W}}(\varrho) &= 0.2 < 0.5 = \mathscr{M}_{\mathscr{W}}(\tau),\\
\mathscr{B}^-_{\mathscr{W}}(\varrho) &= 0.5 > 0.3 = \mathscr{B}^-_{\mathscr{W}}(\tau),\\
\mathscr{B}^+_{\mathscr{W}}(\varrho) &= 0.55 < 0.85 = \mathscr{B}^+_{\mathscr{W}}(\tau),\\
\mathscr{J}_{\mathscr{W}}(\varrho) &= 0.6 > 0.5 = \mathscr{J}_{\mathscr{W}}(\tau).
\end{aligned} \tag{15}$$

Hence, $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ is not a BMBJ-N $\mathbb{H}$-BCK-ideal of $\widetilde{\mathscr{H}}$.

Theorem 3. *In a $\mathbb{H}$-BCK-algebra, every BMBJ-N s-$\mathbb{WH}$-BCK-ideal is a BMBJ-N $\mathbb{WH}$-BCK-ideal.*

Proof. Let $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ be a BMBJ-N s-$\mathbb{WH}$-BCK-ideal of over $\widetilde{\mathscr{H}}$ and let ϱ,τ,u_1,v_1,u $2,v_2,a,b \in \widetilde{\mathscr{H}}$. Then, there exist $z_1 \in \varrho \diamond \tau$, $z_2 \in u_1 \diamond v_1$, $z_3 \in u_2 \diamond v_2$, and $z_4 \in a \diamond b$ such that $\mathscr{M}_{\mathscr{W}}(\varrho) \ge \min\{\mathscr{M}_{\mathscr{W}}(z_1), \mathscr{M}_{\mathscr{W}}(\tau)\}$, $\mathscr{B}^-_{\mathscr{W}}(u_1) \le \max\{\mathscr{B}^-_{\mathscr{W}}(z_2), \mathscr{B}^-_{\mathscr{W}}(v_1)\}$, $\mathscr{B}^+_{\mathscr{W}}(u_2) \ge \min\{\mathscr{B}^+_{\mathscr{W}}(z_3), \mathscr{B}^+_{\mathscr{W}}(v_2)\}$, and $\mathscr{J}_{\mathscr{W}}(a) \le \max$

Table 3: Tabular representation of $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$.

$\mathscr{W}$	$\mathscr{M}_{\mathscr{W}}$	$\widetilde{\mathscr{B}}_{\mathscr{W}}$	$\mathscr{J}_{\mathscr{W}}$
0	0.6	[0.2, 0.95]	0.3
ϱ	0.2	[0.5, 0.55]	0.6
τ	0.5	[0.3, 0.85]	0.5

$\{\mathscr{J}_{\mathscr{W}}(z_4), \mathscr{J}_{\mathscr{W}}(b)\}$ by the condition (ii) of Proposition 2. Since $\mathscr{M}_{\mathscr{W}}(z_1) \geq \inf\{\mathscr{M}_{\mathscr{W}}(z_2) | z_2 \in \varrho \diamond \tau\}$, $\mathscr{B}^-_{\mathscr{W}}(z_2) \leq \sup\{\mathscr{B}^-_{\mathscr{W}}(z_3) | z_3 \in u_1 \diamond v_1\}$, $\mathscr{B}^+_{\mathscr{W}}(z_3) \geq \inf\{\mathscr{B}^+_{\mathscr{W}}(z_4) | z_4 \in u_2 \diamond v_2\}$, and $\mathscr{J}_{\mathscr{W}}(z_4) \leq \sup\{\mathscr{J}_{\mathscr{W}}(z_5) | z_5 \in a \diamond b\}$, it follows that

$$\begin{aligned}
\mathscr{M}_{\mathscr{W}}(\varrho) &\geq \min\{\inf\{\mathscr{M}_{\mathscr{W}}(z_2) | z_2 \in \varrho \diamond \tau\}, \mathscr{M}_{\mathscr{W}}(\tau)\},\\
\mathscr{B}^-_{\mathscr{W}}(u_1) &\leq \max\{\sup\{\mathscr{B}^-_{\mathscr{W}}(z_3) | z_3 \in u_1 \diamond v_1\}, \mathscr{B}^-_{\mathscr{W}}(v_1)\},\\
\mathscr{B}^+_{\mathscr{W}}(u_2) &\geq \min\{\inf\{\mathscr{B}^+_{\mathscr{W}}(z_4) | z_4 \in u_2 \diamond v_2\}, \mathscr{B}^+_{\mathscr{W}}(v_2)\},\\
\mathscr{J}_{\mathscr{W}}(a) &\leq \max\{\sup\{\mathscr{J}_{\mathscr{W}}(z_5) | z_5 \in a \diamond b\}, \mathscr{J}_{\mathscr{W}}(b)\}.
\end{aligned} \tag{16}$$

Therefore, $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ is a BMBJ-N $\mathbb{WH}$-BCK-ideal of over $\widetilde{\mathscr{H}}$. □

Theorem 4. *Let $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ be a BMBJ-N $\mathbb{WH}$-BCK-ideal of $\widetilde{\mathscr{H}}$ which satisfies condition (1) of Proposition 2. Then, $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ is a BMBJ-N s-$\mathbb{WH}$-BCK-ideal of $\widetilde{\mathscr{H}}$.*

Proof. For any $\varrho, \tau, u_1, v_1, u_2, v_2, a, b \in \widetilde{\mathscr{H}}$, there exist ${}_{r^\circ} \in \varrho \diamond \tau$, ${}_{s^\circ} \in u_1 \diamond v_1$, $s_\circ \in u_2 \diamond v_2$, and ${}_{t^\circ} \in a \diamond b$ such that $\mathscr{M}_{\mathscr{W}}({}_{r^\circ}) = \inf\{\mathscr{M}_{\mathscr{W}}(r) | r \in \varrho \diamond \tau\}$, $\mathscr{B}^-_{\mathscr{W}}({}_{s^\circ}) = \sup\{\mathscr{B}^-_{\mathscr{W}}(s) | s \in u_1 \diamond v_1\}$, $\mathscr{B}^+_{\mathscr{W}}(s_\circ) = \inf\left\{\mathscr{B}^+_{\mathscr{W}}(\acute{s}) | \acute{s} \in u_2 \diamond v_2\right\}$, and $\mathscr{J}_{\mathscr{W}}({}_{t^\circ}) = \sup\{\mathscr{J}_{\mathscr{W}}(t) | t \in a \diamond b\}$. Then,

$$\begin{aligned}
\mathscr{M}_{\mathscr{W}}(\varrho) &\geq \min\{\inf\{\mathscr{M}_{\mathscr{W}}(r) | r \in \varrho \diamond \tau\}, \mathscr{M}_{\mathscr{W}}(\tau)\}\\
&= \min\{\mathscr{M}_{\mathscr{W}}({}_{r^\circ}), \mathscr{M}_{\mathscr{W}}(\tau)\},\\
\mathscr{B}^-_{\mathscr{W}}(u_1) &\leq \max\{\sup\{\mathscr{B}^-_{\mathscr{W}}(s) | s \in u_\diamond v_1\}, \mathscr{B}^-_{\mathscr{W}}(v_1)\}\\
&= \max\{\mathscr{B}^-_{\mathscr{W}}({}_{s^\circ}), \mathscr{B}^-_{\mathscr{W}}(v_1)\},\\
\mathscr{B}^+_{\mathscr{W}}(u_2) &\geq \min\left\{\inf\left\{\mathscr{B}^+_{\mathscr{W}}(\acute{s}) | \acute{s} \in u_2 \diamond v_2\right\}, \mathscr{B}^+_{\mathscr{W}}(v_2)\right\}\\
&= \min\left\{\mathscr{B}^+_{\mathscr{W}}\left(\acute{s}_\circ\right), \mathscr{B}^+_{\mathscr{W}}(v_2)\right\},\\
\mathscr{J}_{\mathscr{W}}(a) &\leq \max\{\sup\{\mathscr{J}_{\mathscr{W}}(t) | t \in a \diamond b\}, \mathscr{J}_{\mathscr{W}}(b)\}\\
&= \max\{\mathscr{J}_{\mathscr{W}}({}_{t^\circ}), \mathscr{J}_{\mathscr{W}}(b)\}.
\end{aligned} \tag{17}$$

Therefore, $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ is a BMBJ-N s-$\mathbb{WH}$-BCK-ideal of over $\widetilde{\mathscr{H}}$. □

Theorem 5. *An MBJ-NS $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ over $\widetilde{\mathscr{H}}$ is a BMBJ-N $\mathbb{WH}$-BCK-ideal of $\widetilde{\mathscr{H}}$ if and only if the nonempty sets $U(\mathscr{M}_{\mathscr{W}}, \alpha)$, $L(\mathscr{B}^-_{\mathscr{W}}, \delta^-)$, $U(\mathscr{B}^+_{\mathscr{W}}, \delta^+)$, and $L(\mathscr{J}_{\mathscr{W}}, \gamma)$ are $\mathbb{WH}$-BCK-ideals of $\widetilde{\mathscr{H}}$ for all $\alpha, \delta^-, \delta^-, \gamma \in [0,1]$.*

Proof. It is similar to the proof of Theorem 1. □

The following definition presents the BMBJ-N $\mathscr{S}\mathbb{H}$-BCK-ideal of a $\mathbb{H}$-BCK-algebra $\widetilde{\mathscr{H}}$. Next, we study some properties of this concept.

Definition 5. An MBJ-NS $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ over $\widetilde{\mathscr{H}}$ is called a BMBJ-N $\mathscr{S}\mathbb{H}$-BCK-ideal of $\widetilde{\mathscr{H}}$, if it satisfies

(1) $(\forall \varrho, \tau \in \widetilde{\mathscr{H}})$ $(\inf\{\mathscr{M}_{\mathscr{W}}(z_1) | z_1 \in \varrho \diamond \varrho\} \geq \mathscr{M}_{\mathscr{W}}(\varrho) \mathscr{M}_{\mathscr{W}}(\varrho) \geq \min\{\sup\{\mathscr{M}_{\mathscr{W}}(z_2) | z_2 \in \varrho \diamond \tau\}, \mathscr{M}_{\mathscr{W}}(\tau)\})$,

(2) $(\forall u_1, v_1 \in \widetilde{\mathscr{H}})$ $(\sup\{\mathscr{B}^-_{\mathscr{W}}(z_2) | z_2 \in u_1 \diamond u_1\} \leq \mathscr{B}^-_{\mathscr{W}}(u_1) \mathscr{B}^-_{\mathscr{W}}(u_1) \leq \max\{\inf\{\mathscr{B}^-_{\mathscr{W}}(z_3) | z_3 \in u_1 \diamond v_1\}, \mathscr{B}^-_{\mathscr{W}}(v_1)\})$,

(3) $(\forall u_2, v_2 \in \widetilde{\mathscr{H}})$ $(\inf\{\mathscr{B}^+_{\mathscr{W}}(z_3) | z_3 \in u_2 \diamond u_2\} \geq \mathscr{B}^+_{\mathscr{W}}(u_2)$ $\mathscr{B}^+_{\mathscr{W}}(u_2) \geq \min\{\sup\{\mathscr{B}^+_{\mathscr{W}}(z_4) | z_4 \in u_2 \diamond v_2\}, \mathscr{B}^+_{\mathscr{W}}(v_2)\})$,

(4) $(\forall w, z \in \widetilde{\mathscr{H}})$ $(\sup\{\mathscr{J}_{\mathscr{W}}(z_4) | z_4 \in w \diamond w\} \leq \mathscr{J}_{\mathscr{W}}(w) \mathscr{J}_{\mathscr{W}}(w) \leq \max\{\inf\{\mathscr{J}_{\mathscr{W}}(z_5) | z_5 \in w \diamond z\}, \mathscr{J}_{\mathscr{W}}(z)\})$.

Example 3. Let $\widetilde{\mathscr{H}} = \{0, \varrho, \tau\}$ be a set with the hyperoperation "$\diamond$," which is given by Table 4.

Then, $\widetilde{\mathscr{H}}$ is a $\mathbb{H}$-BCK-algebra (see [9]). Let $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ be an MBJ-NS over $\widetilde{\mathscr{H}}$ given by Table 5.

It is routine to check that $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ is a BMBJ-N $\mathscr{S}\mathbb{H}$-BCK-ideal of $\widetilde{\mathscr{H}}$.

Proposition 3. *Every BMBJ-N $\mathscr{S}\mathbb{H}$-BCK-ideal $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ over $\widetilde{\mathscr{H}}$, satisfying the following conditions:*

(1) $(\forall \varrho \in \widetilde{\mathscr{H}}) \begin{pmatrix} \mathscr{M}_{\mathscr{W}}(0) \geq \mathscr{M}_{\mathscr{W}}(\varrho), \\ \mathscr{B}^-_{\mathscr{W}}(0) \leq \mathscr{B}^-_{\mathscr{W}}(\varrho), \\ \mathscr{B}^+_{\mathscr{W}}(0) \geq \mathscr{B}^+_{\mathscr{W}}(\varrho), \\ \mathscr{J}_{\mathscr{W}}(0) \leq \mathscr{J}_{\mathscr{W}}(\varrho) \end{pmatrix}$,

(2) $(\forall \varrho, \tau \in \widetilde{\mathscr{H}})\left(\varrho \ll \tau \Rightarrow \begin{pmatrix} \mathscr{M}_{\mathscr{W}}(\varrho) \geq \mathscr{M}_{\mathscr{W}}(\tau), \\ \mathscr{B}^-_{\mathscr{W}}(\varrho) \leq \mathscr{B}^-_{\mathscr{W}}(\tau), \\ \mathscr{B}^+_{\mathscr{W}}(\varrho) \geq \mathscr{B}^+_{\mathscr{W}}(\tau), \\ \mathscr{J}_{\mathscr{W}}(\varrho) \leq \mathscr{J}_{\mathscr{W}}(\tau) \end{pmatrix}\right)$,

(3) $(\forall z, \varrho, \tau \in \widetilde{\mathscr{H}})\left(z \in \varrho \diamond \tau \Rightarrow \begin{pmatrix} \mathscr{M}_{\mathscr{W}}(\varrho) \geq \min\{\mathscr{M}_{\mathscr{W}}(z), \mathscr{M}_{\mathscr{W}}(\tau)\}, \\ \mathscr{B}^-_{\mathscr{W}}(\varrho) \leq \max\{\mathscr{B}^-_{\mathscr{W}}(z), \mathscr{B}^-_{\mathscr{W}}(\tau)\}, \\ \mathscr{B}^+_{\mathscr{W}}(\varrho) \geq \min\{\mathscr{B}^+_{\mathscr{W}}(z), \mathscr{B}^+_{\mathscr{W}}(\tau)\}, \\ \mathscr{J}_{\mathscr{W}}(\varrho) \leq \max\{\mathscr{J}_{\mathscr{W}}(z), \mathscr{J}_{\mathscr{W}}(\tau)\} \end{pmatrix}\right)$.

Proof. (1) Since $0 \in \varrho \diamond \varrho \, \forall \varrho \in \widetilde{\mathscr{H}}$, we have

$$\begin{aligned}
\mathscr{M}_{\mathscr{W}}(0) &\geq \inf\{\mathscr{M}_{\mathscr{W}}(z_1) | z_1 \in \varrho \diamond \varrho\} \geq \mathscr{M}_{\mathscr{W}}(\varrho),\\
\mathscr{B}^-_{\mathscr{W}}(0) &\leq \sup\{\mathscr{B}^-_{\mathscr{W}}(z_2) | z_2 \in \varrho \diamond \varrho\} \leq \mathscr{B}^-_{\mathscr{W}}(\varrho),\\
\mathscr{B}^+_{\mathscr{W}}(0) &\geq \inf\{\mathscr{B}^+_{\mathscr{W}}(z_3) | z_3 \in \varrho \diamond \varrho\} \geq \mathscr{B}^+_{\mathscr{W}}(\varrho),\\
\mathscr{J}_{\mathscr{W}}(0) &\leq \sup\{\mathscr{J}_{\mathscr{W}}(z_4) | z_4 \in \varrho \diamond \varrho\} \leq \mathscr{J}_{\mathscr{W}}(\varrho),
\end{aligned} \tag{18}$$

for all $\varrho \in \widetilde{\mathscr{H}}$.

(2) Let $\varrho, \tau \in \widetilde{\mathscr{H}}$ be such that $\varrho \ll \tau$. Then, $0 \in \varrho \diamond \tau$ and thus $\sup\{\mathscr{M}_{\mathscr{W}}(z_1) | z_1 \in \varrho \diamond \tau\} \geq \mathscr{M}_{\mathscr{W}}(0)$, $\inf\{\mathscr{B}^-_{\mathscr{W}}(z_2) | z_2 \in \varrho \diamond \tau\} \leq \mathscr{B}^-_{\mathscr{W}}(0)$, $\sup\{\mathscr{B}^+_{\mathscr{W}}(z_3) | z_3 \in \varrho \diamond \tau\} \geq \mathscr{B}^+_{\mathscr{W}}(0)$, and

Table 4: Tabular representation of the hyperoperation "$\diamond$."

$\diamond$	0	ϱ	τ
0	$\{0\}$	$\{0\}$	$\{0\}$
ϱ	$\{\varrho\}$	$\{0\}$	$\{\varrho\}$
τ	$\{\tau\}$	$\{\tau\}$	$\{0,\tau\}$

Table 5: Tabular representation of $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \mathscr{B}^{-}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$.

$\mathscr{W}$	$\mathscr{M}_{\mathscr{W}}$	$\tilde{\mathscr{B}}_{\mathscr{W}}$	$\mathscr{J}_{\mathscr{W}}$
0	0.63	$[0.09, 0.75]$	0.30
ϱ	0.43	$[0.17, 0.57]$	0.50
τ	0.32	$[0.29, 0.37]$	0.70

$\inf\{\mathscr{J}_{\mathscr{W}}(z_4) | z_4 \in \varrho \diamond \tau\} \leq \mathscr{J}_{\mathscr{W}}(0)$. It follows from Definition 5 that

$$
\begin{aligned}
\mathscr{M}_{\mathscr{W}}(\varrho) &\geq \min\{\sup\{\mathscr{M}_{\mathscr{W}}(z_1) | z_1 \in \varrho \diamond \tau\}, \mathscr{M}_{\mathscr{W}}(\tau)\} \\
&\geq \min\{\mathscr{M}_{\mathscr{W}}(0), \mathscr{M}_{\mathscr{W}}(\tau)\} \\
&= \mathscr{M}_{\mathscr{W}}(\tau), \\
\mathscr{B}^{-}_{\mathscr{W}}(\varrho) &\leq \max\{\inf\{\mathscr{B}^{-}_{\mathscr{W}}(z_2) | z_2 \in \varrho \diamond \tau\}, \mathscr{B}^{-}_{\mathscr{W}}(\tau)\} \\
&\leq \max\{\mathscr{B}^{-}_{\mathscr{W}}(0), \mathscr{B}^{-}_{\mathscr{W}}(\tau)\} \\
&= \mathscr{B}^{-}_{\mathscr{W}}(\tau), \\
\mathscr{B}^{+}_{\mathscr{W}}(\varrho) &\geq \min\{\sup\{\mathscr{B}^{+}_{\mathscr{W}}(z_3) | z_3 \in \varrho \diamond \tau\}, \mathscr{B}^{+}_{\mathscr{W}}(\tau)\} \\
&\geq \min\{\mathscr{B}^{+}_{\mathscr{W}}(0), \mathscr{B}^{+}_{\mathscr{W}}(\tau)\} \\
&= \mathscr{B}^{+}_{\mathscr{W}}(\tau), \\
\mathscr{J}_{\mathscr{W}}(\varrho) &\leq \max\{\inf\{\mathscr{J}_{\mathscr{W}}(z_4) | z_4 \in \varrho \diamond \tau\}, \mathscr{J}_{\mathscr{W}}(\tau)\} \\
&\leq \min\{\mathscr{J}_{\mathscr{W}}(0), \mathscr{J}_{\mathscr{W}}(\tau)\} \\
&= \mathscr{J}_{\mathscr{W}}(\tau),
\end{aligned}
\tag{19}
$$

i.e., $\mathscr{M}_{\mathscr{W}}(\varrho) \geq \mathscr{M}_{\mathscr{W}}(\tau)$, $\mathscr{B}^{-}_{\mathscr{W}}(\varrho) \leq \mathscr{B}^{-}_{\mathscr{W}}(\tau)$, $\mathscr{B}^{+}_{\mathscr{W}}(\varrho) \geq \mathscr{B}^{+}_{\mathscr{W}}(\tau)$, and $\mathscr{J}_{\mathscr{W}}(\varrho) \leq \mathscr{J}_{\mathscr{W}}(\tau)$ for all $\varrho, \tau \in \tilde{\mathscr{H}}$ with $\varrho \ll \tau$.

(3) Let $z, \varrho, \tau \in \mathscr{H}$ be such that $z \in \varrho \diamond \tau$. Then,

$$
\begin{aligned}
\mathscr{M}_{\mathscr{W}}(\varrho) &\geq \min\{\sup\{\mathscr{M}_{\mathscr{W}}(z_1) | z_1 \in \varrho \diamond \tau\}, \mathscr{M}_{\mathscr{W}}(\tau)\} \\
&\geq \min\{\mathscr{M}_{\mathscr{W}}(z), \mathscr{M}_{\mathscr{W}}(\tau)\}, \\
\mathscr{B}^{-}_{\mathscr{W}}(\varrho) &\leq \max\{\inf\{\mathscr{B}^{-}_{\mathscr{W}}(z_2) | z_2 \in \varrho \diamond \tau\}, \mathscr{B}^{-}_{\mathscr{W}}(\tau)\} \\
&\leq \max\{\mathscr{B}^{-}_{\mathscr{W}}(z), \mathscr{B}^{-}_{\mathscr{W}}(\tau)\}, \\
\mathscr{B}^{+}_{\mathscr{W}}(\varrho) &\geq \min\{\sup\{\mathscr{B}^{+}_{\mathscr{W}}(z_3) | z_3 \in \varrho \diamond \tau\}, \mathscr{B}^{+}_{\mathscr{W}}(\tau)\} \\
&\geq \min\{\mathscr{B}^{+}_{\mathscr{W}}(z), \mathscr{B}^{+}_{\mathscr{W}}(\tau)\}, \\
\mathscr{J}_{\mathscr{W}}(\varrho) &\leq \max\{\inf\{\mathscr{J}_{\mathscr{W}}(z_4) | z_4 \in \varrho \diamond \tau\}, \mathscr{J}_{\mathscr{W}}(\tau)\} \\
&\leq \max\{\mathscr{J}_{\mathscr{W}}(z), \mathscr{J}_{\mathscr{W}}(\tau)\},
\end{aligned}
\tag{20}
$$

for all $z, \varrho, \tau \in \tilde{\mathscr{H}}$ with $z \in \varrho \diamond \tau$. □

Corollary 2. *If $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \tilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ is a BMBJ-N $\mathscr{S}\mathbb{H}$-BCK-ideal over $\tilde{\mathscr{H}}$, then condition (2) of Definition 3 is valid.*

Proof. Note that $\mathscr{M}_{\mathscr{W}}(z) \geq \inf\{\mathscr{M}_{\mathscr{W}}(z) | z \in \varrho \diamond \tau\}$, $\mathscr{B}^{-}_{\mathscr{W}}(z) \leq \sup\{\mathscr{B}^{-}_{\mathscr{W}}(z) | z \in \varrho \diamond \tau\}$, $\mathscr{B}^{+}_{\mathscr{W}}(z) \geq \inf\{\mathscr{B}^{+}_{\mathscr{W}}(z) | z \in \varrho \diamond \tau\}$, and $\mathscr{J}_{\mathscr{W}}(z) \leq \sup\{\mathscr{M}_{\mathscr{W}}(z) | z \in \varrho \diamond \tau\}$ for all $z, \varrho, \tau \in \tilde{\mathscr{H}}$ with $z \in \varrho \diamond \tau$. Hence, condition (2) of Definition 3 follows from Proposition 3(2). □

Theorem 6. *Every BMBJ-N $\mathscr{S}\mathbb{H}$-BCK-ideal is a BMBJ-N $\mathbb{H}$-BCK-ideal.*

Proof. Straightforward. □

The converse of Theorem 6 is not true in general. That is, a BMBJ-N $\mathbb{H}$-BCK-ideal may not be a 7BMBJ-N $\mathscr{S}\mathbb{H}$-BCK-ideal.

Example 4. Let $\tilde{\mathscr{H}} = \{0, \varrho, \tau\}$ be a hyper-BCK-algebra as in Example 1. Let $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \tilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ be an MBJ-NS over $\tilde{\mathscr{H}}$ given by Table 6.

Then, $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \tilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ is a BMBJ-N $\mathbb{H}$-BCK-ideal of $\tilde{\mathscr{H}}$, but it is not a BMBJ-N $\mathscr{S}\mathbb{H}$-BCK-ideal of $\tilde{\mathscr{H}}$, since

$$
\begin{aligned}
\mathscr{M}_{\mathscr{W}}(\tau) = 0.32 < 0.63 = \mathscr{M}_{\mathscr{W}}(\varrho) &= \min\{\sup\{\mathscr{M}_{\mathscr{W}}(z) | z \in \tau \diamond \varrho\}, \mathscr{M}_{\mathscr{W}}(\varrho)\}, \\
\mathscr{B}^{-}_{\mathscr{W}}(\tau) = 0.19 > 0.12 = \mathscr{B}^{-}_{\mathscr{W}}(\varrho) &= \max\{\inf\{\mathscr{B}^{-}_{\mathscr{W}}(z) | z \in \tau \diamond \varrho\}, \mathscr{B}^{-}_{\mathscr{W}}(\varrho)\}, \\
\mathscr{B}^{+}_{\mathscr{W}}(\tau) = 0.21 < 0.51 = \mathscr{B}^{+}_{\mathscr{W}}(\varrho) &= \min\{\sup\{\mathscr{B}^{-}_{\mathscr{W}}(z) | z \in \tau \diamond \varrho\}, \mathscr{B}^{-}_{\mathscr{W}}(\varrho)\}, \\
\mathscr{J}_{\mathscr{W}}(\tau) = 0.39 > 0.32 = \mathscr{J}_{\mathscr{W}}(\varrho) &= \max\{\inf\{\mathscr{J}_{\mathscr{W}}(z) | z \in \tau \diamond \varrho\}, \mathscr{J}_{\mathscr{W}}(\varrho)\}.
\end{aligned}
\tag{21}
$$

Table 6: Tabular representation of $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$.

$\mathscr{W}$	$\mathscr{M}_{\mathscr{W}}$	$\widetilde{\mathscr{B}}_{\mathscr{W}}$	$\mathscr{J}_{\mathscr{W}}$
0	0.63	[0.10, 0.71]	0.21
ϱ	0.63	[0.12, 0.51]	0.32
τ	0.32	[0.19, 0.21]	0.39

Theorem 7. *Let $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ be an MBJ-NS over $\widetilde{\mathscr{H}}$. If $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ is a BMBJ-N $\mathscr{S}\mathbb{H}$-BCK-ideal of $\widetilde{\mathscr{H}}$, then the nonempty sets $U(\mathscr{M}_{\mathscr{W}}, \alpha)$, $L(\mathscr{B}^-_{\mathscr{W}}, \delta^-)$, $U(\mathscr{B}^+_{\mathscr{W}}, \delta^+)$, and $L(\mathscr{J}_{\mathscr{W}}, \gamma)$ are $\mathscr{S}\mathbb{H}$-BCK-ideals of $\widetilde{\mathscr{H}}$ for all $\alpha, \delta^-, \delta^+, \gamma \in [0,1]$.*

Proof. Assume that $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ is an MBJ-NS over $\widetilde{\mathscr{H}}$. Let $\alpha, \delta^-, \delta^+, \gamma \in [0,1]$ be such that $U(\mathscr{M}_{\mathscr{W}}, \alpha)$, $L(\mathscr{B}^-_{\mathscr{W}}, \delta^-)$, $U(\mathscr{B}^+_{\mathscr{W}}, \delta^+)$, and $L(\mathscr{J}_{\mathscr{W}}, \gamma)$ are nonempty sets. Then, there exist $r \in U(\mathscr{M}_{\mathscr{W}}, \alpha)$, $t_1 \in L(\mathscr{B}^-_{\mathscr{W}}, \delta^-)$, $t_2 \in U(\mathscr{B}^+_{\mathscr{W}}, \delta^+)$, and $s \in L(\mathscr{J}_{\mathscr{W}}, \gamma)$, and so $\mathscr{M}_{\mathscr{W}}(r) \geq \alpha$, $\mathscr{B}^-_{\mathscr{W}}(t_1) \leq \delta^-$, $\mathscr{B}^+_{\mathscr{W}}(t_2) \geq \delta^+$, and $\mathscr{J}_{\mathscr{W}}(s) \leq \gamma$. Clearly, $0 \in U(\mathscr{M}_{\mathscr{W}}, \alpha)$, $0 \in L(\mathscr{B}^-_{\mathscr{W}}, \delta^-)$, $0 \in U(\mathscr{B}^+_{\mathscr{W}}, \delta^+)$, and $0 \in L(\mathscr{J}_{\mathscr{W}}, \gamma)$ by Proposition 3(1). Now, let $\varrho, \tau, u_1, v_1, u_2, v_2, a, b \in \widetilde{\mathscr{H}}$ be such that $\tau \in U(\mathscr{M}_{\mathscr{W}}, \alpha)$, $(\varrho \diamond \tau) \cap U(\mathscr{M}_{\mathscr{W}}, \alpha) \neq \phi$, $v_1 \in L\ (\mathscr{B}^-_{\mathscr{W}}, \delta^-)$, $(u_1 \diamond v_1) \cap L\ (\mathscr{B}^-_{\mathscr{W}}, \delta^-) \neq \phi$, $v_2 \in U(\mathscr{B}^+_{\mathscr{W}}, \delta^+)$, $(u_2 \diamond v_2) \cap L(\mathscr{B}^+_{\mathscr{W}}, \delta^+) \neq \phi$, $b \in L(\mathscr{J}_{\mathscr{W}}, \gamma)$, and $(a \diamond b) \cap L(\mathscr{J}_{\mathscr{W}}, \gamma) \neq \phi$. Then, there exist $r^\circ \in (\varrho \diamond \tau) \cap U(\mathscr{M}_{\mathscr{W}}, \alpha)$, $t^\circ \in (u_1 \diamond v_1) \cap L(\mathscr{B}^-_{\mathscr{W}}, \delta^-)$, $t_\circ \in (u_2 \ \diamond v_2)\ \cap U(\mathscr{B}^+_{\mathscr{W}}, \delta^+)$, and $s^\circ \in (a \diamond b) \cap L(\mathscr{J}_{\mathscr{W}}, \gamma)$. Hence, $\mathscr{M}_{\mathscr{W}}(r^\circ) \geq \alpha$, $\mathscr{B}^-_{\mathscr{W}}(t^\circ) \leq \delta^-$, $\mathscr{B}^+_{\mathscr{W}}(t_\circ) \geq \delta^+$, and $\mathscr{J}_{\mathscr{W}}(s^\circ) \leq \gamma$. It follows that

$$
\begin{aligned}
\mathscr{M}_{\mathscr{W}}(\varrho) &\geq \min\{\sup\{\mathscr{M}_{\mathscr{W}}(r) | r \in \varrho \diamond \tau\}, \mathscr{M}_{\mathscr{W}}(\tau)\} \\
&\geq \min\{\mathscr{M}_{\mathscr{W}}(_{r^\circ}), \mathscr{M}_{\mathscr{W}}(\tau)\} \\
&\geq \alpha, \\
\mathscr{B}^-_{\mathscr{W}}(u_1) &\leq \max\{\inf\{\mathscr{B}^-_{\mathscr{W}}(t_1) | t_1 \in u_1 \diamond v_1\}, \mathscr{B}^-_{\mathscr{W}}((v_1)\} \\
&\leq \max\{\mathscr{B}^-_{\mathscr{W}}(_{t^\circ}), \mathscr{B}^-_{\mathscr{W}}(v_1)\} \\
&\leq \delta^-, \\
\mathscr{B}^+_{\mathscr{W}}(u_2) &\geq \min\{\sup\{\mathscr{B}^+_{\mathscr{W}}(\acute{t}_1) | \acute{t}_1 \in u_2 \diamond v_2\}, \mathscr{B}^+_{\mathscr{W}}(v_2)\} \\
&\geq \min\left\{\mathscr{B}^+_{\mathscr{W}}(\acute{t}_\circ), \mathscr{B}^+_{\mathscr{W}}(v_2)\right\} \\
&\geq \delta^+, \\
\mathscr{J}_{\mathscr{W}}(a) &\leq \max\{\inf\{\mathscr{J}_{\mathscr{W}}(s) | s \in a \diamond b\}, \mathscr{J}_{\mathscr{W}}(b)\} \\
&\leq \max\{\mathscr{J}_{\mathscr{W}}(_{s^\circ}), \mathscr{J}_{\mathscr{W}}(b)\} \\
&\leq \gamma.
\end{aligned}
\tag{22}
$$

Hence, $\varrho \in U(\mathscr{M}_{\mathscr{W}}, \alpha)$, $u_1 \in L(\mathscr{B}^-_{\mathscr{W}}, \delta^-)\, u_2 \in L\ (\mathscr{B}^+_{\mathscr{W}}, \delta^+)$, and $a \in L(\mathscr{J}_{\mathscr{W}}, \gamma)$. Therefore, $U(\mathscr{M}_{\mathscr{W}}, \alpha)$, $L(\mathscr{B}^-_{\mathscr{W}}, \delta^-)$, $U(\mathscr{B}^+_{\mathscr{W}}, \delta^+)$, and $L(\mathscr{J}_{\mathscr{W}}, \gamma)$ are $\mathscr{S}\mathbb{H}$-BCK-ideals of $\widetilde{\mathscr{H}}$. □

Theorem 8. *Let $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ be an MBJ-NS over $\widetilde{\mathscr{H}}$ which satisfies the following condition:*

$$
(\forall \mathscr{K} \subseteq \widetilde{\mathscr{H}})\left(\exists \varrho_\circ, _{\tau^\circ}, _{\xi^\circ}, _{\eta^\circ} \in \mathscr{K}\right) \text{ such that } \left(
\begin{aligned}
\mathscr{M}_{\mathscr{W}}\left(\varrho_\circ\right) &= \sup\{\mathscr{M}_{\mathscr{W}}(\varrho) | \varrho \in \mathscr{K}\} \\
\mathscr{B}^-_{\mathscr{W}}\left(_{\tau^\circ}\right) &= \inf\{\mathscr{B}^-_{\mathscr{W}}(\tau) | \tau \in \mathscr{K}\} \\
\mathscr{B}^+_{\mathscr{W}}\left(_{\xi^\circ}\right) &= \sup\{\mathscr{B}^+_{\mathscr{W}}(\xi) | \tau \in \mathscr{K}\} \\
\mathscr{J}_{\mathscr{W}}\left(_{\eta^\circ}\right) &= \inf\{\mathscr{J}_{\mathscr{W}}(\eta) | \eta \in \mathscr{K}\}
\end{aligned}
\right).
\tag{23}
$$

If the nonempty sets $U(\mathscr{M}_{\mathscr{W}}, \alpha)$, $L(\mathscr{B}^-_{\mathscr{W}}, \delta^-)$, $U(\mathscr{B}^+_{\mathscr{W}}, \delta^+)$, and $L(\mathscr{J}_{\mathscr{W}}, \gamma)$ are $\mathscr{S}\mathbb{H}$-BCK-ideals of $\widetilde{\mathscr{H}}$ for all $\alpha, \delta^-, \delta^+, \gamma \in [0,1]$, then $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \widetilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ is a BMBJ-N $\mathscr{S}\mathbb{H}$-BCK-ideal of $\widetilde{\mathscr{H}}$.

Proof. Suppose the nonempty sets $U(\mathscr{M}_{\mathscr{W}}, \alpha)$, $L(\mathscr{B}^-_{\mathscr{W}}, \delta^-)$, $U(\mathscr{B}^+_{\mathscr{W}}, \delta^+)$, and $L(\mathscr{J}_{\mathscr{W}}, \gamma)$ are $\mathscr{S}\mathbb{H}$-BCK-ideals of $\widetilde{\mathscr{H}}$ for all $\alpha, \delta^-, \delta^+, \gamma \in [0,1]$. Then, $\varrho \in U(\mathscr{M}_{\mathscr{W}}, \alpha)$, $\tau \in L(\mathscr{B}^-_{\mathscr{W}}, \delta^-)$, $\xi \in U(\mathscr{B}^+_{\mathscr{W}}, \delta^+)$, and $\eta \in L(\mathscr{J}_{\mathscr{W}}, \gamma)$ for some $\varrho, \tau, \xi, \eta \in \widetilde{\mathscr{H}}$, and so $\varrho \diamond \varrho \ll \{\varrho\} \subseteq U(\mathscr{M}_{\mathscr{W}}, \alpha)$, $\tau \diamond \tau \ll \{\tau\} \subseteq L(\mathscr{B}^-_{\mathscr{W}}, \delta^-)$, $\xi \diamond \xi \ll \{\xi\} \subseteq U(\mathscr{B}^+_{\mathscr{W}}, \delta^+)$, and $\eta \diamond \eta \ll \{\eta\} \subseteq L(\mathscr{J}_{\mathscr{W}}, \gamma)$. By Lemma 1, we have $\varrho \diamond \varrho \subseteq U(\mathscr{M}_{\mathscr{W}}, \alpha)$, $\tau \diamond \tau \subseteq L(\mathscr{B}^-_{\mathscr{W}}, \delta^-)$, $\xi \diamond \xi \subseteq U(\mathscr{B}^+_{\mathscr{W}}, \delta^+)$, and $\eta \diamond \eta \subseteq L(\mathscr{J}_{\mathscr{W}}, \gamma)$. Thus, for any $a \in \varrho \diamond \varrho, b \in \tau \diamond \tau$, $c \in \xi \diamond \xi$, and $d \in \eta \diamond \eta$, we get $a \in U(\mathscr{M}_{\mathscr{W}}, \alpha), b \in L(\mathscr{B}^-_{\mathscr{W}}, \delta^-), c \in U(\mathscr{B}^+_{\mathscr{W}}, \delta^+)$, and $d \in L(\mathscr{J}_{\mathscr{W}}, \gamma)$. Hence, $\mathscr{M}_{\mathscr{W}}(a) \geq \alpha, \mathscr{B}^-_{\mathscr{W}}(b) \leq \delta^-, \mathscr{B}^+_{\mathscr{W}}(c) \geq \delta^+$, and $\mathscr{J}_{\mathscr{W}}(d) \leq \gamma$. It follows that

$$
\begin{aligned}
\inf\{\mathscr{M}_{\mathscr{W}}(a) | a \in \varrho \diamond \varrho\} &\geq \alpha = \mathscr{M}_{\mathscr{W}}(\varrho), \\
\sup\{\mathscr{B}^-_{\mathscr{W}}(b) | b \in \tau \diamond \tau\} &\leq \delta^- = \mathscr{B}^-_{\mathscr{W}}(\tau), \\
\inf\{\mathscr{B}^+_{\mathscr{W}}(c) | c \in \xi \diamond \xi\} &\geq \delta^+ = \mathscr{B}^+_{\mathscr{W}}(\xi), \\
\sup\{\mathscr{J}_{\mathscr{W}}(d) | d \in \eta \diamond \eta\} &\leq \gamma = \mathscr{J}_{\mathscr{W}}(\eta).
\end{aligned}
\tag{24}
$$

For any $\varrho, \tau, u_1, v_1, u_2, v_2, w, z \in \widetilde{\mathscr{H}}$, taking $r = \min\{\sup\{\mathscr{M}_{\mathscr{W}}(a) | a \in \varrho \diamond \tau\}, \mathscr{M}_{\mathscr{W}}(\tau)\}$, $t = \max\{\inf\{\mathscr{B}^-_{\mathscr{W}}(b) | b \in u_1 \diamond v_1\}, \mathscr{B}^-_{\mathscr{W}}(v_1)\}$, $t = \min\{\sup\{\mathscr{B}^+_{\mathscr{W}}(c) | c \in u_2 \diamond v_2\}, \mathscr{B}^+_{\mathscr{W}}(v_2)\}$, and $s = \max\{\inf\{\mathscr{J}_{\mathscr{W}}(d)\ | d \in w \diamond z\}, \mathscr{J}_{\mathscr{W}}(z)\}$. Then, by assumption, $U(\mathscr{M}_{\mathscr{W}}, \alpha)$, $L(\mathscr{B}^-_{\mathscr{W}}, \delta^-)$, $U(\mathscr{B}^+_{\mathscr{W}}, \delta^+)$, and $L(\mathscr{J}_{\mathscr{W}}, \gamma)$ are strong hyper-BCK-ideals of $\widetilde{\mathscr{H}}$. Condition (3) implies that there exist $a^\circ \in \varrho \diamond \tau, b^\circ \in u_1 \diamond v_1$, $c^\circ \in u_2 \diamond v_2$, and $d^\circ \in w \diamond z$ such that $\mathscr{M}_{\mathscr{W}}(_{a^\circ}) = \sup\{\mathscr{M}_{\mathscr{W}}(a) | a \in \varrho \diamond \tau\}, \mathscr{B}^-_{\mathscr{W}}(b^\circ) = \inf\{\mathscr{B}^-_{\mathscr{W}}(b) | b \in u_1 \diamond v_1\}$, $\mathscr{B}^+_{\mathscr{W}}(_{c^\circ}) = \sup\{\mathscr{B}^+_{\mathscr{W}}(c) | c \in u_2 \diamond v_2\}$, and $\mathscr{J}_{\mathscr{W}}(d^\circ) = \sup\{\mathscr{J}_{\mathscr{W}}(d) | d \in w \diamond z\}$. Hence,

$$\begin{aligned}
\mathscr{M}_{\mathscr{W}}\left(_{a^{\circ}}\right) &= \sup\{\mathscr{M}_{\mathscr{W}}(a)|a \in \varrho \diamond \tau\} \geq \min\{\sup\{\mathscr{M}_{\mathscr{W}}(a)|a \in \varrho \diamond \tau\}, \mathscr{M}_{\mathscr{W}}(\tau)\} = r,\\
\mathscr{B}_{\mathscr{W}}^{-}\left(_{b^{\circ}}\right) &= \inf\{\mathscr{B}_{\mathscr{W}}^{-}(b)|b \in u_1 \diamond v_1\} \leq \max\{\inf\{\mathscr{B}_{\mathscr{W}}^{-}(b)|b \in u_1 \diamond v_1\}, \mathscr{B}_{\mathscr{W}}^{-}(v_1)\} = t,\\
\mathscr{B}_{\mathscr{W}}^{+}\left(_{c^{\circ}}\right) &= \sup\{\mathscr{B}_{\mathscr{W}}^{+}(c)|c \in u_2 \diamond v_2\} \geq \min\{\sup\{\mathscr{B}_{\mathscr{W}}^{+}(c)|c \in u_2 \diamond v_2\}, \mathscr{B}_{\mathscr{W}}^{+}(v_2)\} = t',\\
\mathscr{J}_{\mathscr{W}}\left(_{d^{\circ}}\right) &= \inf\{\mathscr{J}_{\mathscr{W}}(d)|d \in w \diamond z\} \leq \max\{\inf\{\mathscr{J}_{\mathscr{W}}(d)|d \in w \diamond z\}, \mathscr{M}_{\mathscr{W}}(z)\} = s.
\end{aligned} \tag{25}$$

This imply that $a^{\circ} \in U(\mathscr{M}_{\mathscr{W}}, \alpha)$, $b^{\circ} \in L(\mathscr{B}_{\mathscr{W}}^{-}, \delta^{-})$, $c^{\circ} \in U(\mathscr{B}_{\mathscr{W}}^{+}, \delta^{+})$, and $d^{\circ} \in L(\mathscr{J}_{\mathscr{W}}, \gamma)$. Hence, $(\varrho \diamond \tau) \cap U(\mathscr{M}_{\mathscr{W}}, r) \neq \phi$, $(u_1 \diamond v_1) \cap L(\mathscr{B}_{\mathscr{W}}^{-}, t) \neq \phi$, $(u_2 \diamond v_2) \cap U(\mathscr{B}_{\mathscr{W}}^{+}, t) \neq \phi$, $(w \diamond z) \cap L(\mathscr{J}_{\mathscr{W}}, s) \neq \phi$, and thus $\varrho \in U(\mathscr{M}_{\mathscr{W}}, r)$, $u_1 \in L(\mathscr{B}_{\mathscr{W}}^{-}, t)\, u_2 \in U(\mathscr{B}_{\mathscr{W}}^{+}, t)$, and $w \in L(\mathscr{J}_{\mathscr{W}}, s)$. It follows that

$$\begin{aligned}
\mathscr{M}_{\mathscr{W}}(\varrho) &\geq r = \min\{\sup\{\mathscr{M}_{\mathscr{W}}(a)|a \in \varrho \diamond \tau\}, \mathscr{M}_{\mathscr{W}}(\tau)\},\\
\mathscr{B}_{\mathscr{W}}^{-}(u_1) &\leq t = \max\{\inf\{\mathscr{B}_{\mathscr{W}}^{-}(b)|b \in u_1 \diamond v_1\}, \mathscr{B}_{\mathscr{W}}^{-}((v_1)\},\\
\mathscr{B}_{\mathscr{W}}^{+}(u_2) &\geq t' = \min\{\sup\{\mathscr{B}_{\mathscr{W}}^{+}(c)|c \in u_2 \diamond v_2\}, \mathscr{B}_{\mathscr{W}}^{+}(v_2)\},\\
\mathscr{J}_{\mathscr{W}}(d) &\leq s = \max\{\inf\{\mathscr{J}_{\mathscr{W}}((d)|d \in w \diamond z\}, \mathscr{J}_{\mathscr{W}}(z)\}.
\end{aligned} \tag{26}$$

Therefore, $\mathscr{W} = (\mathscr{M}_{\mathscr{W}}, \tilde{\mathscr{B}}_{\mathscr{W}}, \mathscr{J}_{\mathscr{W}})$ is a BMBJ-N $\mathscr{S}\mathbb{H}$-BCK-ideal of $\tilde{\mathscr{H}}$. □

4. Conclusions

This paper found a new link between hyperalgebraic structures and MBJ-NSs by introducing BMBJ-N $\mathbb{WH}$-BCK-ideal, BMBJ-N s-$\mathbb{WH}$-BCK-ideal, and BMBJ-N $\mathscr{S}\mathbb{H}$-BCK-ideal of $\mathbb{H}$-BCK-algebras and studying their properties and relations. BMBJ-N $\mathbb{WH}$-BCK-ideals and BMBJ-N $\mathscr{S}\mathbb{H}$-BCK-ideals in relation to level subsets have been discussed. Conditions for a BMBJ-N $\mathbb{WH}$-BCK-ideal to be a BMBJ-N s-$\mathbb{WH}$-BCK-ideal have been found. Conditions for an MBJ-NS to be a BMBJ-N $\mathscr{S}\mathbb{H}$-BCK-ideal have been given. The results in this paper can be considered as a generalization of the results known for (intuitionistic) fuzzy hyperideals of $\mathbb{H}$-BCK-algebras. In future work, various types of BMBJ-N $\mathbb{H}$-BCK-ideals will be defined and discussed.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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