ISSN: 2582-0818

Volume 3, Number 4 (2020)

Annals of Communications in Mathematics

An International Journal

http://www.technoskypub.com/journal/acm/

Annals of Communications in Mathematics

Volume 3, Number 4 (2020)

Table of Contents

Articles

ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 3, Number 4 (2020), 242-251
ISSN: 2582-0818

THE THIRD ISOMORPHISM THEOREM FOR (CO-ORDERED) Γ-SEMIGROUPS WITH APARTNESS

DANIEL A. ROMANO

ABSTRACT. The notion of Γ-semigroups has been introduced by M. K. Sen and N. K. Saha. The concept of (co-ordered) Γ-semigroups with apartness in Bishop's constructive algebra was introduced by this author. Many classical notions and processes of semigroups and Γ-semigroups have been extended to (co-ordered) Γ-semigroups with apartness such as ideals, filters and the first theorem of isomorphism of this class of algebraic structures. In this paper, as a continuation of earlier research, the author designs a form of the third isomorphism theorems for Γ-semigroups and co-ordered Γ-semigroups with apartness which does not have its counterpart in the classical Γ-semigroup theory.

1. INTRODUCTION

The principle-logical framework of this article is Bishop constructive mathematics ([1, 2, 8]) which implies intuitionistic logic ([16]).

Within this framework the author is interested in Γ-semigroups with apartness as a continuation of his research [9, 11, 12, 13]. The concept of the Γ-semigroup with apartness was introduced in the article [9]. Semilattice congruences in Γ-semigroup with apartness were the focus of the paper [11] while the article [12] analyzes some substructures of co-ordered Γ-semigroup with apartnss such as co-filters. In article [13], two forms of the first theorem on isomorphism between (co-ordered) Γ-semigroups with apartness are considered.

In this report, as a continuation of previous research, the author presents a form of the third theorem on isomorphism between (co-ordered) Γ-semigroups with apartness which does not have its counterpart in the classical Γ-semigroup theory: Theorem 3.2 for Γ-semigroup with apartness and Theorem 3.4 for co-ordered Γ-semigroup with apartness.

The concept of co-congruences on Γ-semigroup with apartness was introduced and analyzed in the author's article [9]. The presence of a Γ -cocongruence q on a (co-ordered) Γ-semigroup with apartness $(S, =, \neq, w)$ it allows the construction of a (co-ordered) Γ - semigroup with apartness $[S : q] = \{xq : x \in S\}$. This type of (co-ordered) Γ-semigroups has no counterpart in the classical theory of Γ-semigroups. Let q_1 and q_2 be co-congruences on a Γ-semigroup with apartness S such that $q_2 \subseteq q_1$. Theorem 3.2

2010 *Mathematics Subject Classification.* 03F65, 06F05, 06F99, 20M99.
Key words and phrases. Bishop's constructive algebra, Intuitionistic logic, Γ-semigroup with apartness, (co-ordered) Γ-semigroup with apartness, co-congruence in Γ-semigroup, the third isomorphism theorem.
Received: October 10, 2020. Accepted: December 01, 2020. Published: December 31, 2020.

shows the relationship between Γ-semigroups $[S : q_1]$ and $[S : q_2]$. Theorem 3.4 describes an analogous relationship between Γ-semigroups $[S : q_1]$ and $[S : q_2]$ if Γ-semigroup with apartness S is ordered with respect to two co-orders relations.

2. Preliminaries

The concept of Γ-semigroup was developed in the papers [14, 15]. Articles [3, 4] study isomorphisms on (ordered) Γ-semigroups.

2.1. **Γ-semigroup with apartness.** To explain the notions and notations used in this article, which but not previously described, we instruct the reader to look at the articles [3, 4, 6, 7, 14, 15]. Here we will introduce some specific notions and substructures of this semigroups that appear only in the **Bish** version by reference to the author's reports [9, 10, 11, 12, 13].

Definition 2.1. ([9], Definition 2.1) Let $(S, =_S, \neq_S)$ and $(\Gamma, =_\Gamma, \neq_\Gamma)$ be two non-empty sets with apartness. Then S is called a Γ-*semigroup with apartness* if there exist a strongly extensional total mapping

$$w_S : S \times \Gamma \times S \ni (x, a, y) \longmapsto w_S(x, a, y) := xay \in S$$

satisfying the condition

$$(\forall x, y, z \in S)(\forall a, b \in \Gamma)((xay)bz =_S xa(ybz)).$$

We recognize immediately that the following implication

$$(\forall x, y, u, v \in S)(\forall a, b \in \Gamma)(xay \neq_S ubv \implies (x \neq_S u \lor a \neq_\Gamma b \lor y \neq_S v))$$

is valid, because w_S is a strongly extensional function.

Example 2.2. Let $\mathbb{N}$ be a semiring of natural numbers, $\mathbb{R}$ be the filed of real number, where the apartness is determined as follows

$$(\forall x, y \in \mathbb{R})(x \neq y \iff (\exists k \in \mathbb{N})(|x - y| > \frac{1}{k})),$$

$S := [0, 1] \subseteq \mathbb{R}$ and $\Gamma := \{\frac{1}{n} : n \in \mathbb{N}\}$. Then S is a commutative Γ-semigroup under the usual multiplication.

Example 2.3. Let S be the set $M_{2\times3}(\mathbb{R})$ of all 2×3 matrices over the set of real numbers $\mathbb{R}$ and Γ be the set $M_{3\times2}(\mathbb{R})$ of all 3×2 matrices over $\mathbb{R}$. Apartness '$\neq$' is defined in $M_{2\times3}(\mathbb{R})$ in the standard way

$$(\forall A, B \in M_{2\times3}(\mathbb{R}))(A \neq B \iff (\exists (i, j) \in \{1, 2\} \times \{1, 2, 3\})(a_{ij} \neq b_{ij})).$$

The apartness relation in $M_{3\times2}(\mathbb{R})$ is defined analogously. Define $A\alpha B =$ usual matrix product of A, α, B for all $A, B \in S$ and for all $\alpha \in \Gamma$. Then S is a Γ-semigroup with apartness. Note that S is not a semigroup.

Definition 2.4. ([9], Definition 2.2) Let S be a Γ-semigroup with apartness. A subset T of S is said to be a Γ-*cosubsemigroup* of S if the following holds

$$(\forall x, y \in S)(\forall a \in \Gamma)(xay \in T \implies (x \in T \lor y \in T)).$$

We will assume that the empty set $\emptyset$ is a Γ-cosubsemigroup of a Γ-semigroup S by definition.

2.2. **Co-ordered Γ-semigroup with apartness.** The relation α is said to be a co-order on the set $(X, =_X, \neq_X)$ with apartness if it is consistent $\alpha \subseteq \neq_X$, co-transitive $\alpha \subseteq \alpha * \alpha$, i.e.

$$(\forall x, y, z \in X)((x, z) \in \alpha \implies ((x, y) \in \alpha \vee (y, z) \in \alpha))$$

and linear in the following sense: $\neq_X \subseteq \alpha \cup \alpha^{-1}$. α is said to be a co-quasiorder on X if it is a consistent and co-transitive relation on X.

In order to avoid misunderstanding the notations used, we remind yourself that equality and apartness are determined on the product $S \times T$ in the following way

$$(\forall x, y \in S)(\forall u, v \in T)((x, u) = (y, v) \iff (x =_S y \wedge u =_T v))$$

and

$$(\forall x, y \in S)(\forall u, v \in T)((x, u) \neq (y, v) \iff (x \neq_S y \vee u \neq_T v)).$$

A brief recapitulation of a number of algebraic structures ordered by co-quasiorder relation is presented in the review paper [10].

In the following definition we introduce the concept of co-order relations in Γ-semigroup with apartnesst.

Definition 2.5. ([11], Definition 3.1) Let S be a Γ-semigroup with apartness. A co-order relation $\nleqslant_S$ on S is *compatible* with the semigroup operations in S if the following holds

$$(\forall x, y, z \in S)(\forall a \in \Gamma)((xaz \nleqslant_S yaz \vee zax \nleqslant_S zay) \implies x \nleqslant_S y).$$

In this case it is said that S is an *ordered* Γ-semigroup *under co-order* $\nleqslant_S$ or it is *co-ordered* Γ-semigroup.

If we speak the language of classical algebra, then the relation $\nleqslant_S$ is compatible with the operation w_S in S if this operation is cancellative with respect to the co-order.

Example 2.6. Let $M = \{a, b, c\}$, where is it $a \neq b$, $b \neq c$ and $a \neq c$, and $\Gamma = \{\gamma\}$ with the internal operation w_M defined as in [5], Example 1.11. Then M is a commutative Γ-semigroup. Define a relation $\nleqslant$ on M as follows $\nleqslant := \{(b, a), (b, c), (c, a), (c, b)\}$. Then M is a co-ordered set. It can be verified that M is a ordered Γ-semigroup under the co-order relation $\nleqslant$.

Example 2.7. Let $\mathbb{N}$ be additive semigroup of natural numbers. If we put $S := (\mathbb{N}, =, \neq, +, \nleqslant)$, $\Gamma := \{5\}$, where an apartness $\neq$ is determined by $x \neq y \iff \neg(x = y)$, and $w(x, 5, y) = x + 5 + y \in \mathbb{N}$, then S is ordered Γ-semigroup under the co-order $\nleqslant$ determined by $x \nleqslant y \iff \neg(x \leqslant y)$.

Example 2.8. Let S be a set of all reverse isotone se-mapping from an ordered set $(P, =, \neq)$ under a co-order $\nleqslant_P$ into another ordered set $(Q, =, \neq)$ under the co-order $\nleqslant_Q$ and let Γ be set of all reverse isotone se-mapping from Q to P. In both cases, apartness is defined as follows $f \neq g \iff (\exists x)(f(x) \neq g(x))$. For $f, g \in S$ and $\alpha \in \Gamma$ put $f \alpha g = g \circ \alpha \circ f$, where $' \circ '$ be mark for standard composition between relations. Then S is a Γ-semigroup. A co-order $' \nleqslant '$ relation on S can defined by $f \nleqslant g \iff (\exists x \in P)(f(x) \nleqslant_Q g(x))$. Then S is a co-ordered Γ-semigroup.

2.3. **Γ-cocongruences.** The notion of the co-equality relation in sets with apartness introduced and analyzed by this author (see, for example, [10]). The relation q is a co-equality on a set $(S, =_S, \neq_S)$ if it is a consistent, symmetric and co-transitive relation on S. A co-congruence on some algebraic structure $((S, =_S, \neq_S), \cdot)$ is a coequality relation on S which is compatible in one very specific sense with the internal operation in S. A reader can look at these specific features in the author review article [10]. A co-equality relation q on a semigroup with apartness S is a co-congruence in S if the following holds

$$(\forall x, y, u, v \in S)((xu, yv) \in q \implies ((x, y) \in q \vee (u, v)).$$

The concept of Γ-cocongruence on Γ-semigroups with apartness was introduced in [9] by the following definition

Definition 2.9. ([9], Definition 2.6) Let S be a Γ-semigroup with apartness. A co-equality relation $q \subseteq S \times S$ is called a Γ - *cocongruence* on S if the following holds

$$(xau, ybv) \in q \implies ((x, y) \in q \vee a \neq_\Gamma b \vee (u, v) \in q)$$

for any $x, y, u, v \in S$ and all $a, b \in \Gamma$.

Let q be a co-congruence on Γ-semigroup with apartness S. This tool it allows the construction of the congruence relation $q^\lhd$ on S compatible with q. The pair $(q^\lhd, q)$ it allows to design Γ-semigroup with apartness $S/(q^\lhd, q)$. In addition to this structure, the relation q it allows the design of Γ-semigroup with apartness $[S : q] := \{xq : x \in S\}$ which has no counterpart in the classical Γ-semigroup theory.

In this paper, the focus is on this structure and in what follows, some of the properties of this last algebraic structure are analyzed.

2.4. **Homomorphism between Γ-semigroups.** In this subsection, the determination of the concept of Γ-homomorphism between Γ-semigroups with apartness is taken from article [9].

Definition 2.10. ([9], Definition 2.7) Let $(S, =_S, \neq_S)$ is a Γ-semigroup and $(T, =_T, \neq_T)$ a Λ - semigroups with apartness. A pair (h, φ) of strongly extensional functions $h : S \longrightarrow T$ and $\varphi : \Gamma \longrightarrow \Lambda$ is called a *se-homomorphism* from Γ-semigroup S to Λ-semigroup T if the following holds

$$(\forall x, y \in S)(\forall a \in \Gamma)((h, \varphi)(xay) =_T h(x)\varphi(a)h(y)).$$

The following can be verified without difficulty.

Lemma 2.1. *Let (h, φ) be a se-homomorphism from Γ-semigroup with apartness S to a Λ-semigroup with apartness R. Then holds*

$$(h, \varphi) \circ w_S = w_T \circ (h, \varphi, h)$$

where (h, φ, h) is understood as follows:

$$(\forall x, y \in S)(\forall a \in \Gamma)((h, \varphi, h)(x, a, y) := (h(x), \varphi(a), h(y))).$$

Of course, the specificity of this determination is in the requirement that the functions $h : S \longrightarrow T$ and $\varphi : \Gamma \longrightarrow \Lambda$ must be strongly extensional functions. It is easily verified that (h, φ) is a correctly determined strongly extensive function.

Also, it is easy to see that:

Proposition 2.2. *Let $(h, \varphi) : S \longrightarrow T$ be a se-homomorphism. Then the relation*

$$q := Coker(h, \varphi) := \{(x, y) : (h, \varphi)(x) \neq_T (h, \varphi)(y)\}$$

is a Γ-cocongruence on S.

3. The Main Results

In this section, which is the central part of this paper, we are talking about 'the Third theorem on isomorphism between Γ-semigroups with apartness' and 'the Third theorem on isomorphism between ordered Γ-semigroup with apartness under a co-order. The specificity of this presentation is that the forms of this theorem that do not have their counterparts in the classical Γ-semigroup theory.

3.1. **The case of Γ-semigroup with apartness.** Let q_1 and q_2 be co-congruences on a Γ-semigroup with apartness $(S, =_S, \neq_S, w_S)$ such that $q_2 \subseteq q_1$. We can construct Γ-semigroups with apartness $([S : q_1], =_1, \neq_1, w_{[S:q_1]})$ and $([S : q_2], =_2, \neq_2, w_{[S:q_2]})$. In the following theorem we will give a construction of relation $[q_2 : q_1]$ on $[S : q_1]$ by relations q_1 and q_2.

Theorem 3.1. *Let q_1 and q_2 be co-congruences on a Γ-semigroup with apartness S such that $q_2 \subseteq q_1$. Then the relation $[q_2 : q_1] \subseteq [S : q_1] \times [S : q_1]$, defined by*

$$(\forall xq_1, yq_1 \in [S : q_1])((xq_1, yq_1) \in [q_2 : q_1] \iff (x, y) \in q_2),$$

is a co-congruence on $[S : q_1]$.

Proof. We will first show that $[q_2 : q_1]$ is a co-equality relation on $[S : q_1]$. Let $x, y, z \in S$ be elements such that $(xq_1, zq_1) \in [q_2 : q_1]$. Then

(i) $(xq_1, zq_1) \in [q_2 : q_1] \iff (x, z) \in q_2$
$$\implies ((x, y) \in q_2 \vee (y, z) \in q_2)$$
$$\implies ((xq_1, yq_1) \in [q_2 : q_1] \vee (yq_1, zq_1) \in [q_2 : q_1]).$$

(ii) $(xq_1, yq_1) \in [q_2 : q_1] \iff (x, y) \in q_2$
$$\iff (y, x) \in q_2$$
$$\iff (yq_1, xq_1) \in [q_2 : q_1].$$

(iii) $(xq_1, yq_1) \in [q_2 : q_1] \implies (x, y) \in q_2 \subseteq q_1$
$$\iff xq_1 \neq_1 yq_1.$$

Second, let us check that $[q_2 : q_1]$ is a co-congruence on $[S : q_1]$. Take $x, y, u, v \in S$ and $a, b \in \Gamma$ such that $((xq_1)a(yq_1), (uq_1)b(vq_1)) \in [q_2 : q_1]$. Then

$$\begin{aligned} ((xay)q_1, (ubv)q_1) \in [q_2 : q_1] &\iff (xay, ubv) \in q_2 \\ &\implies (x, u) \in q_2 \subseteq q_1 \vee a \neq_\Gamma b \vee (y, v) \in q_2 \subseteq q_1 \\ &\implies xq_1 \neq_1 yq_1 \vee a \neq_\Gamma b \vee yq_1 \neq_1 vq_1. \end{aligned}$$ □

We can ([13], Theorem 3.3) design Γ-semigroup

$$[[S; q_1] : [q_2 : q_1]] := \{(xq_1)[q_2 : q_1] : xq_1 \in [S : q_1]\}$$

with

$$(\forall x, y \in S)((xq_1)[q_2 : q_1] =_3 (yq_1)[q_2 : q_1] \iff (xq_1, yq_1) \lhd [q_2 : q_1])$$

and

$$(\forall x, y \in S)((xq_1)[q_2 : q_1] \neq_3 (yq_1)[q_2 : q_1] \iff (xq_1, yq_1) \in [q_2 : q_1]),$$

where the internal operation

$w_3 := w_{[[S:q_1]:[q_2:q_1]]} : [[S; q_1 : [q_2 : q_1]] \times \Gamma \times [[S : q_1] : [q_2 : q_1]] \longrightarrow [[S : q_1] : [q_2 : q_1]]$

is determined as follows

$$w_3((xq_1)[q_2 : q_1], a, (yq_1)[q_2 : q_1]) := ((xay)q_1)[q_2 : q_1].$$

Also, according to Theorem 3.3 in [13], there are unique se-epimorphisms

$$\vartheta_1 : S \longrightarrow [S : q_1],\ \vartheta_2 : S \longrightarrow [S : q_2] \text{ and } \vartheta_{21} : [S : q_1] \longrightarrow [[S : q_1] : [q_2 : q_1]].$$

In order to be able to design the third isomorphism theorem for this class of γ-semigroups with apartness we need the following lemma

One form of the third isomorphism theorem between Γ-semigroups with apartness that does not have its counterpart in the classical theory of Γ-semigroups can now be designed.

Theorem 3.2. *Let q_1 and q_2 be co-congruences on a Γ-semigroup with apartness S such that $q_2 \subseteq q_1$. Then there is a unique injective, embedding and surjective se-homomorphism $\beta : [S : q_1] : [q_2 : q_1]] \longrightarrow [S : q_2]$ such that $\vartheta_2 = \beta \circ \vartheta_{21} \circ \vartheta_1$.*

Proof. Define $\beta : [[S : q_1] : [q_2 : q_1]] \longrightarrow [S : q_2]$ by

$$(\forall x[q_2 : q_1] \in [[S : q_1] : [q_2 : q_1]])(\beta((xq_1)[q_2 : q_1]) := xq_2).$$

(a) First, let's show that β is well-defined se-mapping.

Let $x, y, u, v \in S$ be such that $(xq_1)[q_2 : q_1] =_3 (yq_1)[q_2 : q_1]$ and $(uq_1, vq_1) \in [q_2 : q_1]$. Then $(xq_1, yq_1) \lhd [q_2 : q_1]$ and $(u, v) \in q_2$. Thus

$$\begin{aligned}
&(uq_1, vq_1) \in [q_2 : q_1] \implies \\
&\qquad (uq_1, xq_1) \in [q_2 : q_1] \lor (xq_1, yq_1) \in [q_2 : q_1] \lor (yq_1, vq_1) \in [q_2 : q_1] \\
&\qquad \implies (u, x) \in q_2 \lor (y, v) \in q_2 \\
&\qquad \implies u \neq_S x \lor y \neq_S v \\
&\qquad \implies (x, y) \neq (u, v) \in q_2.
\end{aligned}$$

This means $(x, y) \lhd q_2$. Thus

$$\beta((xq_1)[q_2 : q_1]) := xq_2 =_2 yq_2 := \beta((yq_1)[q_2 : q_1]).$$

On the other hand, if

$$\beta((xq_1)[q_2 : q_1]) \neq_2 \beta((yq_1)[q_2 : q_1]),$$

we have $xq_2 \neq_2 yq_2$. Then $(x, y) \in q_2$. Thus $(xq_1, yq_1) \in [q_2 : q_1]$. So,

$$(xq_1)[q_2 : q_1] \neq_3 (yq_1)[q_2 : q_1].$$

(b) Second, it should be shown that β is an injective mapping. Let $x, y, u, v \in S$ be such that

$$\beta((xq_1)[q_2 : q_1]) =_2 \beta((yq_1)[q2 : q_1])$$

and $(uq_1, vq_1) \in [q_2 : q_1]$. This means $xq_2 =_2 yq_2$ and $(u, v) \in q_2$. Then $(x, y) \lhd q_2$. Further on, from $(u, v) \in q_2$ we have

$$\begin{aligned}
(u, v) \in q_2 &\implies (u, x) \in q_2 \subseteq q_1 \lor (x, y) \in q_2 \lor (y, v) \in q_2 \subseteq q_1 \\
&\implies uq_1 \neq_1 xq_1 \lor yq_1 \neq vq_1 \\
&\implies (xq_1, yq_1) \neq (uq_1, vq_1) \in [q_2 : q_1].
\end{aligned}$$

This means $(xq_1, yq_1) \lhd [q_2 : q_1]$. Hence

$$(xq_1)[q_2 : q_1] =_3 (yq_1)[q_2 : q_1].$$

(c) Let us prove that β is an embedding. Let $x, y \in S$ be elements such that $(xq_1)[q_2 : q_1] \neq_3 (yq_1)[q_2 : q_1]$. Then $(xq_1, yq_1) \in [q_2 : q_1]$. So, $(x, y) \in q_2$. Hence $xq_2 \neq_2 yq_2$. This means $\beta((xq_1)[q_2 : q_1]) \neq_2 \beta((yq_1)[q_2 : q_1])$.

(d) Since it is obvious that β is surjective, it remains to show that (β, i) is a homomorphism of Γ-semigroups. For $x, y \in S$ and $a \in \Gamma$, we have

$$\begin{aligned}
&(\beta, i)(((xq_1)[q_2 : q_1])\, a\, ((yq_1)[q_2 : q_1])) =_2 \\
&(\beta, i)(w_3((xq_1)[q_2 : q_1],\, a,\, (yq_1)[q_2 : q_1])) =_2 (\beta, i)((((xay)q_1)[q_2 : q_1]) \\
&=_2 (xay)q_2 =_2 w_2(xq_2,\, a,\, yq_2) =_2 w_2(\beta((xq_1)[q_2 : q_1]),\, a,\, \beta((yq_1)[q_2 : q_1]))
\end{aligned}$$

$=_2 (\beta((xq_1)[q_2 : q_1))a(\beta((yq_1)[q_2 : q_1])).$

(e) Finally, we show that the required equality is valid. For arbitrary $x \in S$ we have

$$\begin{aligned}\vartheta_2(x) &:= xq_2 =_2 \beta((xq_1)[q_2 : q_1]) \\ &=_2 \beta(\vartheta_{21}(xq_1)) =_2 \beta(\vartheta_{21}(\vartheta_1(x))) \\ &=_2 (\beta \circ \vartheta_{21} \circ \vartheta_1)(x).\end{aligned}$$

□

3.2. **The case of co-ordered Γ-semigroup with apartness.** Let us consider ordered Γ-semigroup with apartness $(S, =_S, \neq_S, \nleqslant_S)$ under a co-order $\nleqslant_S$ and let σ and τ be co-quasiorder relation on S such that $\sigma \subseteq \tau \subseteq \nleqslant_S$ and suppose that σ and τ satisfy the condition of Definition 2.5:

$$(\forall x, y, z \in S)(\forall a \in \Gamma))(((xaz, yaz) \in \sigma \vee (zax, zay) \in \sigma) \implies (x, y) \in \sigma),$$

$$(\forall x, y, z \in S)(\forall a \in \Gamma))(((xaz, yaz) \in \tau \vee (zax, zay) \in \tau) \implies (x, y) \in \tau).$$

It is known that co-congruences $q_\sigma = \sigma \cup \sigma^{-1}$ and $q_\tau = \tau \cup \tau^{-1}$ on S such that $q_\sigma \subseteq q_\tau$ can be designed. Further, by Theorem 4.3 in [13], this allows us to construct Γ-semigroups with apartness $([S : q_\sigma], =_\sigma, \neq_\sigma, w_\sigma)$ and $([S : q_\tau], =_\tau, \neq_\tau, w_\tau)$ which are ordered by co-order relations $\nleqslant_\sigma$ and $\nleqslant_\tau$ respectfully as follows

$$(\forall x, y \in S)(xq_\sigma \nleqslant_\sigma yq_\sigma \iff (x, y) \in \sigma)$$

and

$$(\forall x, y \in S)(xq_\tau \nleqslant_\tau yq_\tau \iff (x, y) \in \tau).$$

Let us define the relation $[\sigma : \tau]$ on co-ordered Γ-semigroup with apartness $[S : \tau]$ as follows

$$(\forall x, y \in S)((xq_\tau, yq_\tau) \in [\sigma : \tau] \iff (x, y) \in \sigma).$$

Theorem 3.3. *Let σ and τ be co-quasiorder on a co-ordered Γ-semigroup with apartness $(S, =_S, \neq_S, \nleqslant_S)$ with the internal operation $w_S : S \times \Gamma \times S \longrightarrow S$ such that $\sigma \subseteq \tau \subseteq \nleqslant_S$. Then $[\sigma : \tau]$ is a co-quasiorder relation on $[S : q_\tau]$ compatible with the internal operation w_τ.*

Proof. Ler $x, y, z \in S$ be arbitrary elements. Then:

$$\begin{aligned}(xq_\tau, yq_\tau) \in [\sigma : \tau] &\implies (x, y) \in \sigma \subseteq \tau \subseteq q_\tau \\ &\implies xq_\tau \neq_\tau yq_\tau; \\ (xq_\tau, zq_\tau) \in [\sigma : \tau] &\iff (x, z) \in \sigma \\ &\implies (x, y) \in \sigma \vee (y, z) \in \sigma \\ &\implies (xq_\tau, yq_\tau) \in [\sigma : \tau] \vee (yq_\tau, zq_\tau) \in [\sigma : \tau].\end{aligned}$$

Let us show that this relation is compatible with the internal operation w_τ in $[S : q_\tau]$. For $x, y, z \in S$ we have

$$\begin{aligned}((xaz)q_\tau, (yaz)q_\tau) \in [\sigma : \tau] &\iff (xaz, yaz) \in \sigma \\ &\implies (x, y) \in \sigma \\ &\iff (xq_\tau, yq_\tau) \in [\sigma : \tau].\end{aligned}$$

The second implication can be proved in an analogous way. □

For ease of writing, let's put

$$q := q_{[\sigma:\tau]} = [\sigma : \tau] \cup [\sigma : \tau]^{-1}.$$

Without major difficulties it can be verified that q is a co-congruence on the co-ordered Γ-semigroup with apartness $([S, q_\tau], =_\tau, \neq_\tau, \nleqslant_\tau)$ compatible with its internal operation

w_τ. According to the previous theorem, Theorem 3.3, Γ-semigroup with apartness

$$([[S : q_\tau] : q], =_3, \neq_3, \not\leq_3, w_3)$$

can be designed, where is

$$(\forall x, y \in S)((xq_\tau)q =_3 (yq_\tau)q \iff (xq_\tau, yq_\tau) \lhd q),$$
$$(\forall x, y \in S)((xq_\tau)q \neq_3 (yq_\tau)q \iff (xq_\tau, yq_\tau) \in q).$$

The internal operation w_3 in $[[S; q_\tau] : q]$ is determined as follows

$$w_3((xq_\tau)q,\ a\,(yq_\tau)q) := (xq_\tau)q \cdot a \cdot (yq_\tau)q := ((xay)q_\tau)q$$

for any $(xq_\tau)q, (yq_\tau)q \in [[S : q_\tau] : q]$ and $a \in \Gamma$.
The co-order relation $\not\leq_3$ in $[[S : q_\tau] : q]$ is determined as follows

$$(\forall x, y \in S)((xq_\tau)q \not\leq_3 (yq_\tau)q \iff (xq_\tau, yq_\tau) \in [\sigma : \tau]).$$

We can now design and prove the following theorem which we recognize as 'The third isomorphism theorem between co-ordered Γ-semigroups with apartness'. Of course, this form of this theorem does not have its counterpart in the classical Γ-semigroup theory.

Theorem 3.4. *Let σ and τ be co-quasiorder relations on co-ordered Γ-semigro- up with apartness $(S, =_S, \neq_S, \not\leqslant_S, w_S)$ such that $\sigma \subseteq \tau \subseteq \not\leqslant_S$. Then there is a unique injective, embedding and surjective se-homomorphism*

$$\gamma : [[S; q_\tau] : q] \longrightarrow [S; q_\sigma].$$

Proof. Let us define γ by

$$(\forall (xq_\tau)q \in [[S : q_\tau] : q])(\gamma((xq_\tau)q) := xq_\sigma).$$

First, let us show that γ is a well-defined mapping. Assume $x, y, u, v \in S$ are such that $(xq_\tau)q =_3 (yq_\tau)q$ and $(u, v) \in q_\sigma$. then $(xq_\tau, yq_\tau) \lhd q$. On the other hand, from $(u, v) \in q_\sigma$ we get $(u, x) \in q_\sigma \vee (x, y) \in q_\sigma \vee (y, v) \in q_\sigma$. If we assume that $(x, y) \in q_\sigma$ is valid, then we would have the following $(x, y) \in \sigma$ or $(y, x) \in \sigma$. It would follow from here

$$(x, y) \in \sigma \vee (y, x) \in \sigma \implies (xq_\tau, yq_\tau) \in [\sigma : \tau] \subseteq q \vee (yq_t, xq_t) \in [\sigma : \tau] \subseteq q$$

which would contradict the hypothesis $(xq_\tau, yq_\tau) \lhd q$. So, it has to be $(u, x) \in q_\sigma$ or $(y, v) \in q_\sigma$. Thus $x \neq_S u$ or $y \neq_S v$. Therefore, $(x, y) \neq (u, v) \in q_\sigma$. This means $(x, y) \lhd q_\sigma$. Hence

$$\gamma((xq_\tau)q) := xq_\sigma =_\sigma yq_\sigma := \gamma((yq_\tau)q).$$

Second, let us show that γ is a se-mapping. Let $x, y \in S$ be such that

$$xq_\sigma = \gamma((xq_\tau)q) \neq_3 \gamma((yq_\tau)q) = yq_\sigma.$$

Then $(x, y) \in q_\sigma = \sigma \cup \sigma^{-1}$. Thus $((xq_\tau, yq_\tau) \in [\sigma : \tau] \cup [\sigma : \tau]^{-1} = q$. Hence

$$(xq_\tau)q \neq_3 (yq_\tau)q.$$

Let $x, y, u, v \in S$ be such that $xq_\sigma =_\sigma yq_\sigma$ and $(uq_\tau, vq_\tau) \in q$. Then $(x, y) \lhd q_\sigma = \sigma \cup \sigma^{-1}$ and $(uq_\tau, vq_\tau) \in q$. Thus $(uq_\tau, xq_\tau) \in q$ or $(xq_\tau, yq_\tau) \in q$ or $(yq_\tau, vq_{tau}) \in q$. The option $(xq_\tau, yq_\tau) \in q$ gives $(x, y) \in \sigma \cup \sigma^{-1}$ which is in contradiction with the hypothesis. So, it has to be $(uq_\tau, xq_\tau) \in q$ or $(yq_\tau, vq_\tau) \in q$. Thus $xq_\tau \neq_\tau uq_\tau$ or $yq_\tau \neq_\tau vq_\tau$. Hence $(xq_\tau, yq_\tau) \neq (uq_\tau, vq_\tau) \in q$. This means $(xq_\tau, yq_\tau) \lhd q$, i.e. $(xq_\tau)q =_3 (yq_\tau)q$. This shows that γ is an injective mapping.

It remains to show that γ is an embedding. Let $x, y \in S$ be such that $(xq_\tau)q \neq_3 (yq_\tau)q$. Then $(xq_\tau, yq_\tau) \in q = [\sigma : \tau] \cup [\sigma : \tau]^{-1}$. Thus $(x, y) \in \sigma \cup \sigma^{-1} = q_\sigma$. Hence $xq_\sigma \neq_\sigma yq_\sigma$.

Finally, it is obvious that γ is a surjective mapping. □

4. Final comments

'Why could this selected material be of interest to the mathematical public?" — It's a very pertinent question that someone can ask. A significant number of members of this community do not fully understand some of the claims made in this and similar papers, and often the concepts and claims in them are thought of as something that should not deserve the attention of that community. The following could serve as a kind of justification:

Within the chosen principled-logical environment, the notions and processes with them treated in this and similar articles are logically possible.

New techniques have been designed and they have proven successful for processing settings in such a chosen work environment.

This enriches mathematics, our understanding of structures in algebra, but also opens new perspectives in the perception of observed algebraic structures.

Last but not least, this report presents the algebraic concepts Γ-semigroups with apartness and processes with them such as se-isomorphisms between them, which readers cannot encounter in the classical algebra.

In research what follows, the author will try to complete the research on Γ-semigroups with apartness by establishing of dual of (ordered) ideals and their properties in such algebraic structures.

5. Acknowledgements

The author is obliged the thankful to the reviewer for the numerous and significant suggestions that raised the consistency of the ideas presented in this paper.

References

[1] E. Bishop. *Foundations of Constructive Analysis*. New York: McGraw-Hill, 1967.
[2] D. S. Bridges and F. Richman. *Varieties of Constructive Mathematics*. Cambridge: London Mathematical Society Lecture Notes, No. 97, Cambridge University Press, 1987.
[3] R. Chinram and K. Tinpun. Isomorphism theorems for Γ-semigroups and ordered Γ-semigroups. *Thai Journal of Mathematics*, **7**(2)(2009), 231-241.
[4] H. Hedayati. Isomorphisms via congruences on Γ-semigroups and Γ-ideals. *Thai J. Math.*, 11(3)(2013), 563–575.
[5] K. Hila. Filters in ordered Γ-semigroups. *Rocky. Moun. J. Math.*, **41**(1)(2011), 189–203.
[6] N. Kehayopulu. On ordered Γ-semigroups. *Sci. Math. Japonicae Online*, **e-2010**, 37-43.
[7] A. Mahnoob and N. M. Khan. Generalized fuzzy Γ-ideals of ordered Γ-semigroups. *Annals of Communications in Mathematics*, **2**(2)(2019), 91–100.
[8] R. Mines, F. Richman and W. Ruitenburg. *A course of constructive algebra*. Springer-Verlag, New York 1988.
[9] D. A. Romano. Γ-semigroups with apartness. *Bull. Allahabad Math. Soc.*, **34**(1)(2019), 71–83.
[10] D. A. Romano. Some algebraic structures with apartness, A review. *J. Int. Math. Virtual Inst.*, **9**(2)(2019), 361–395.
[11] D. A. Romano. Semilattice co-congruence in Γ-semigroups. *Turkish Journal of Mathematics and Computer Science*, **12**(1)(2020), 1–7.
[12] D. A. Romano. Co-filters in Γ-semigroups ordered under co-order. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi, Ser. Nouă, Mat.*, **66**(2)(2020), 161–168.
[13] D. A. Romano. The first isomorphism theorem for (co-ordered) Γ-semigroup with apartness. (To appear) Available at 'Researchgate'.

[14] M. K Sen. On Γ-semigroups. In *Proceeding of International Conference on 'Algebra and its Applications,(New Delhi, 1981)'* (pp. 301–308). Lecture Notes in Pure and Appl. Math. 9, New York: Decker Publication, 1984.

[15] M. K. Sen and N. K. Saha. On Γ-semigroup, I. *Bull. Calcutta Math. Soc.*, **78**(1986), 181–186.

[16] A. S. Troelstra and D. van Dalen. *Constructivism in Mathematics: An Introduction.* Amsterdam: North-Holland, 1988.

DANIEL A. ROMANO
INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE,
6, KORDUNAŠKA STREET, 78000 BANJA LUKA, BOSNIA AND HERZEGOVINA
Email address: bato49@hotmail.com

ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 3, Number 4 (2020), 252-260
ISSN: 2582-0818

FUZZY STABILITY RESULTS DERIVING FROM QUADRATIC FUNCTIONAL EQUATION

K. TAMILVANAN, G. BALASUBRAMANIAN AND K. LOGANATHAN*

ABSTRACT. We establish the generalized Hyers-Ulam stability of the 4 variable quadratic functional equation

$$\begin{aligned}\phi(2v_1 \pm v_2 \pm v_3 \pm v_4) &= \phi(v_1 \pm v_3 \pm v_4) + \phi(v_1 \pm v_2 \pm v_3) + \phi(v_1 \pm v_2 \pm v_4) \\ &\quad + \phi(\pm v_1) - \phi(\pm v_2) - \phi(\pm v_3) - \phi(\pm v_4)\end{aligned}$$

in Fuzzy Normed Spaces using two different methods.

1. INTRODUCTION AND PRELIMINARIES

Stability of some functional equations within the framework of fuzzy normed spaces or random normed spaces has been investigated. While is it true that a function which about fulfills a functional equation, Hyers theorem become generalized via Aoki for additive mappings and through Rassias for linear mappings with the aid of thinking about an unbounded Cauchy difference. Eventually, the stability problems for several sorts of functional equations in numerous spaces were appreciably studied by using many authors [1, 3, 4, 6, 8, 9, 10, 11, 13, 15, 16, 17, 18].

The well-known functional equation inside the field of stability of functional equation is the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.1}$$

The function $f(x) = x^2$ is the solution of the functional equation (1.1).

A generalization of the Rassias theorem turned into acquired by means of Gavruta through changing the unbounded difference via a wellknown control function the purpose of this paper is to have a look at the opportunity of changing the most powerful norm within the essential theorem, with an arbitrary continuous norm, so that it will attain a stability end result for fuzzy normed spaces over discipline with valuation Rassias changed into the primary to show that there exists a completely unique linear mapping fulfilling all through the actually field. Numerous outcomes for the Hyers-Ulam-Rassias stability of many functional equations had been proved via numerous researchers [2, 5, 7, 12, 14, 19].

2010 *Mathematics Subject Classification.* 39B52, 39B72, 39B82.
Key words and phrases. Quadratic Functional Equation; Fixed Point; Hyers-Ulam Stability; Fuzzy normed Space.
Received: October 11, 2020. Accepted: December 03, 2020. Published: December 31, 2020.
*Corresponding author.

In this current work, authors establish the Hyers-Ulam stability results of the 4 variable functional equation

$$\phi\left(2v_1 \pm v_2 \pm v_3 \pm v_4\right) = \phi\left(v_1 \pm v_3 \pm v_4\right) + \phi\left(v_1 \pm v_2 \pm v_3\right) + \phi\left(v_1 \pm v_2 \pm v_4\right) \\ +\phi\left(\pm v_1\right) - \phi\left(\pm v_2\right) - \phi\left(\pm v_3\right) - \phi\left(\pm v_4\right) \qquad (1.2)$$

in Fuzzy normed spaces using direct and fixed point methods.

Definition 1.1. Let M be a real linear space. A function $H : M \times R \to [0, 1]$ is said to be fuzzy norm on M if for all $v, w \in M$ and $a, b \in R$.
(N_1) $H(v, a) = 0$ for $a \leq 0$;
(N_2) $v = 0$ iff $H(v, a) = 1$ for all $a > 0$;
(N_3) $H(av, b) = H\left(v, \frac{b}{|a|}\right)$ if $a \neq 0$;
(N_4) $H(v + w, a + b) \geq min\{H(v, a), H(w, b)\}$;
(N_5) $H(v, \cdot)$ is a non-decreasing function on R and $\lim_{a\to\infty} H(v, a) = 1$.
(N_6) For $v \neq 0, H(v, \cdot)$ is continuous on R.
The pair (M, H) is called a fuzzy normed linear space.

Theorem 1.1. **[*The Alternative of fixed point*]** *Suppose that for a complete generalized metric space (M, d) and a strictly contractive mapping $T : M \to N$ with Lipschitz constant L. Then, for each given element $v \in M$ either*
(B1) $(T^n v, T^{n+1} v) = +\infty, \quad \forall n \geq 0$, or
(B2) there exists natural number n_0 such that:

i) $d(T^n v, T^{n+1} v) < \infty \quad \forall n \geq n_0$;
ii) The sequence $(T^n v)$ is convergent to a fixed point w^* of T;
iii) w^* is the unique fixed point of T in the set $N = \{w \in M; d(T^{n_0} v, w) < \infty\}$;
iv) $d(w^*, w) \leq \frac{1}{1-L} d(w, Tw) \forall w \in L$.

Throughout the upcoming sections, we consider M, (Z, H') and (N, H) are Linear space, Fuzzy normed space and Fuzzy Banach space respectively. For the notational convenient, we define $\phi : M \to N$ by

$$D\phi(v_1, v_2, v_3, v_4) = \phi\left(2v_1 \pm v_2 \pm v_3 \pm v_4\right) - \phi\left(v_1 \pm v_3 \pm v_4\right) - \phi\left(v_1 \pm v_2 \pm v_3\right) - \phi\left(v_1 \pm v_2 \pm v_4\right) \\ -\phi\left(\pm v_1\right) + \phi\left(\pm v_2\right) + \phi\left(\pm v_3\right) + \phi\left(\pm v_4\right)$$

for all $v_1, v_2, v_3, v_4 \in M$.

2. STABILITY RESULTS FOR (1.2): DIRECT METHOD

Theorem 2.1. *Let $\varphi \in \{-1, 1\}$ be fixed and let $\Upsilon : M^4 \to Z$ be a mapping such that for some $\varsigma > 0$ and $\left(\frac{\varsigma}{5^2}\right)^\varphi < 1$*

$$H'\left(\Upsilon\left(5^\varphi v, 5^\varphi v, 5^\varphi v, 5^\varphi v\right), \tau\right) \geq H'\left(\varsigma^\varphi \Upsilon(v, v, v, v), \tau\right) \quad \forall v \in M, \tau > 0 \qquad (2.1)$$

and

$$\lim_{p\to\infty} H'\left(\Upsilon(5^{\varphi p} v_1, 5^{\varphi p} v_2, 5^{\varphi p} v_3, 5^{\varphi p} v_4), 5^{2\varphi p}\tau\right) = 1$$

for all $v_1, v_2, v_3, v_4 \in M$ and all $\tau > 0$. Suppose an even mapping $\phi : M \to N$ satisfies the inequality

$$H\left(D\phi(v_1, v_2, v_3, v_4), \tau\right) \geq H'\left(\Upsilon(v_1, v_2, v_3, v_4), \tau\right) \qquad (2.2)$$

for all $\tau > 0$ and all $v_1, v_2, v_3, v_4 \in M$. Then the limit

$$Q_2(v) = H - \lim_{p\to\infty} \frac{\phi(5^{\varphi p} v)}{5^{2\varphi p}}$$

exists for all $v \in M$ and the mapping $Q_2 : M \to N$ is a unique quadratic mapping such that

$$H\left(\phi(v) - Q_2(v), \tau\right) \geq H^{'}\left(\Upsilon(v,v,v,v), \tau \mid 5^2 - \varsigma \mid\right) \tag{2.3}$$

for all $v \in M$ and all $\tau > 0$.

Proof. Consider $\varphi = 1$. Replacing (v_1, v_2, v_3, v_4) by (v,v,v,v) in (2.2), we get

$$H\left(\phi(5v) - 25\phi(v), \tau\right) \geq H^{'}\left(\Upsilon(v,v,v,v), \tau\right) \quad \forall v \in M, \tau > 0. \tag{2.4}$$

From that (2.4)

$$H\left(\frac{\phi(5v)}{5^2} - \phi(v), \frac{\tau}{5^2}\right) \geq H^{'}\left(\Upsilon(v,v,v,v), \tau\right) \quad \forall v \in M, \tau > 0. \tag{2.5}$$

Interchanging v by $5^p v$ in (2.5), we obtain

$$H\left(\frac{\phi(5^{p+1}v)}{5^2} - \phi(5^p v), \frac{\tau}{5^2}\right) \geq H^{'}\left(\Upsilon(5^p v, 5^p v, 5^p v, 5^p v), \tau\right) \quad \forall v \in M, \tau > 0. \tag{2.6}$$

Using (2.1), (N_3) in (2.6) we get

$$H\left(\frac{\phi(5^{p+1}v)}{5^2} - \phi(5^p v), \frac{\tau}{5^2}\right) \geq H^{'}\left(\Upsilon(v,v,v,v), \frac{\tau}{\varsigma^p}\right) \quad \forall v \in M, \tau > 0. \tag{2.7}$$

It is easy to show that from (2.7), we have

$$H\left(\frac{\phi(5^{p+1}v)}{5^{2(p+1)}} - \frac{\phi(5^p v)}{5^{2p}}, \frac{\tau}{5^{2(p+1)}}\right) \geq H^{'}\left(\Upsilon(v,v,v,v), \frac{\tau}{\varsigma^p}\right) \tag{2.8}$$

holds for all $v \in M, \tau > 0$. Switching τ through $\varsigma^p \tau$ in (2.8), we have

$$H\left(\frac{\phi(5^{p+1}v)}{5^{2(p+1)}} - \frac{\phi(5^p v)}{5^{2p}}, \frac{\varsigma^p \tau}{5^{2(p+1)}}\right) \geq H^{'}\left(\Upsilon(v,v,v,v), \tau\right) \quad \forall v \in M, \tau > 0. \tag{2.9}$$

It is easy to show that

$$\frac{\phi(5^p v)}{5^{2p}} - \phi(v) = \sum_{l=0}^{p-1} \frac{\phi(5^{l+1}v)}{5^{2(l+1)}} - \frac{\phi(5^l v)}{5^{2l}} \tag{2.10}$$

for all $v \in M$. From (2.9) and (2.10), we have

$$\begin{aligned} H\left(\frac{\phi(5^p v)}{5^{2p}} - \phi(v), \sum_{l=0}^{p-1} \frac{\tau\varsigma^l}{5^{2(l+1)}}\right) &\geq \min\{H\left(\frac{\phi(5^{l+1}v)}{5^{2(l+1)}} - \frac{\phi(5^l v)}{5^{2l}}, \frac{\tau\varsigma^l}{5^{2(l+1)}}\right) \quad : l = 0,1,2,3\} \\ &\geq H^{'}\left(\Upsilon\left(v,v,v,v\right), \tau\right) \quad \forall v \in M, \tau > 0. \end{aligned} \tag{2.11}$$

Interchanging v by $5^q v$ in (2.11) and utilizing (2.1), (N_3), we reach

$$H\left(\frac{\phi(5^{p+q}v)}{5^{2(p+q)}} - \frac{\phi(5^q v)}{5^{2q}}, \sum_{l=0}^{p-1} \frac{\tau\varsigma^l}{5^{2(l+1)}}\right) \geq H^{'}\left(\Upsilon(5^q v, 5^q v, 5^q v, 5^q v), \tau\right)$$

$$\geq H^{'}\left(\Upsilon(v,v,v,v),\tfrac{\tau}{\varsigma^q}\right)$$

and so

$$H\left(\frac{\phi(5^{p+q}v)}{5^{2(p+q)}}-\frac{\phi(5^qv)}{5^{2q}},\sum_{l=q}^{p+q-1}\frac{\tau\varsigma^l}{5^{2(l+1)}}\right)\geq H^{'}\left(\Upsilon(v,v,v,v),\tau\right)\quad\forall v\in M,\tau>0. \tag{2.12}$$

and all $p,q\geq 0$. Replacing τ by $\frac{\tau}{\sum_{l=q}^{p+q-1}\frac{\varsigma^l}{5^{2(l+1)}}}$ in (2.12), we get

$$H\left(\frac{\phi(5^{p+q}v)}{5^{2(p+q)}}-\frac{\phi(5^qv)}{5^{2q}},\tau\right)\geq H^{'}\left(\Upsilon(v,v,v,v),\frac{\tau}{\sum_{l=q}^{p+q-1}\frac{\varsigma^l}{5^{2(l+1)}}}\right)\quad\forall v\in M,\tau>0. \tag{2.13}$$

and all $p,q\geq 0$. As $0<\varsigma<5^2$ and $\sum_{l=0}^{p}\left(\frac{\varsigma}{5^2}\right)^l<\infty$, the Cauchy criterion for convergence and (N_5) implies that $\{\frac{\phi(5^pv)}{5^{2p}}\}$ is a Cauchy sequence in (N,H). Since (N,H) is a fuzzy Banach space, this $\{\frac{\phi(5^pv)}{5^{2p}}\}$ converges to some point $Q_2(v)\in N$. So one can define $Q_2:M\to N$ by

$$Q_2(v):=H-\lim_{p\to\infty}\frac{\phi(5^pv)}{5^{2p}}$$

for all $v\in M$. Since ϕ is even. Letting $p=0$ in (2.13), we obtain

$$H\left(\frac{\phi(5^pv)}{5^{2p}}-\phi(v),\tau\right)\geq H^{'}\left(\Upsilon(v,v,v,v),\frac{\tau}{\sum_{l=0}^{p-1}\frac{\varsigma^l}{5^{2(l+1)}}}\right)\quad\forall v\in M,\tau>0. \tag{2.14}$$

Passing the limit as $p\to\infty$ in (2.14) and utilizing (N_6), we have

$$H\left(\phi(v)-Q_2(v),\tau\right)\geq H^{'}\left(\Upsilon(v,v,v,v),\tau(5^2-\varsigma)\right)$$

for all $v\in M$ and all $\tau>0$. Now we claim that Q_2 is quadratic. Replacing (v_1,v_2,v_3,v_4) by $(5^pv_1,5^pv_2,5^pv_3,5^pv_4)$ in (2.2) respectively, we have

$$H\left(\frac{1}{5^{2p}}D\phi\left(5^pv_1,5^pv_2,5^pv_3,5^pv_4\right),\tau\right)\geq H^{'}\left(\Upsilon(5^pv_1,5^pv_2,5^pv_3,5^pv_4),5^{2p}\tau\right)$$

for all $v\in M$ and all $\tau>0$. Since

$$\lim_{p\to\infty}H^{'}\left(\Upsilon(5^pv_1,5^pv_2,5^pv_3,5^pv_4),5^{2p}\tau\right)=1$$

Q_2 fulfils (1.2). Therefore, $Q_2:M\to N$ is quadratic. Next, to show the uniqueness of Q_2, let R_2 be another quadratic function which fulfilling (2.3). Fix $v\in M$, clearly $Q_2(5^pv)=5^{2p}Q_2(v)$ and $R_2(5^pv)=5^{2p}R_2(v)\quad\forall v\in M,\ p>0$. From (2.3), we have

$$\begin{aligned}H(Q_2(v)-R_2(v),\tau)\qquad &=H\left(\frac{Q_2(5^pv)}{5^{2p}}-\frac{R_2(5^pv)}{5^{2p}},\tau\right)\\&\geq H^{'}\left(\Upsilon(v,v,v,v),\frac{(5^{2p})\tau(5^2-\varsigma)}{2\varsigma^p}\right)\end{aligned}$$

for all $v\in M$ and all $\tau>0$. Since $\lim_{p\to\infty}\frac{(5^{2p})\tau(5^2-\varsigma)}{2\varsigma^p}=\infty$, we have

$$\lim_{p\to\infty}H^{'}\left(\Upsilon(v,v,v,v),\frac{(5^{2p})\tau(5^2-\varsigma)}{2\varsigma^p}\right)=1.$$

Thus $H(Q_2(v)-R_2(v),\tau)=1$ for all $v\in M$ and all $\tau>0$, and so $Q_2(v)=R_2(v)$. In similar manner, we can obtain the stability results for $\varphi=-1$. $\square$

Corollary 2.2. *Suppose that the function $\phi : M \to N$ satisfies the inequality*

$$H(D\phi(v_1, v_2, v_3, v_4), \tau) \geq \begin{cases} H'(\eta, \tau), \\ H'\left(\eta||v_1||^a + ||v_2||^a + ||v_3||^a + ||v_4||^a, \tau\right), \\ H'\Big(\eta\Big(||v_1||^a + ||v_2||^a + ||v_3||^a + ||v_4||^a + \\ \qquad\qquad ||v_1||^{4a} \cdot ||v_2||^{4a} \cdot ||v_3||^{4a} \cdot ||v_4||^{4a}\Big), \tau\Big), \end{cases}$$

for all $v_1, v_2, v_3, v_4 \in M$ and all $\tau > 0$, where η, a are constants with $\eta > 0$. Then there exists a unique quadratic mapping $Q_2 : M \to N$ such that

$$H(\phi(v) - Q_2(v), \tau) \geq \begin{cases} H'(\eta, \mid 24 \mid \tau), \\ H'\left(4\eta||v||^a, \mid 5^2 - 5^a \mid \tau\right); & a \neq 2, \\ H'\left(5\eta||v||^{4a}, \mid 5^2 - 5^{4a} \mid \tau\right); & a \neq \frac{2}{4}, \end{cases}$$

for all $v \in M$ and all $\tau > 0$.

3. Stability results for (1.2):fixed point method

For to prove the stability result, we define the following θ_l is a constant such that

$$\theta_l = \begin{cases} 5 & \text{if} \quad l = 0; \\ \frac{1}{5} & \text{if} \quad l = 1; \end{cases}$$

and χ is the set such that $\chi = \{s/s : M \to N, s(0) = 0\}$.

Theorem 3.1. *Let $\phi : M \to N$ be a mapping for which there exists a function $\Upsilon : M^4 \to N$ with condition*

$$\lim_{p\to\infty} H^{'}\left(\Upsilon(\theta^p v_1, \theta^p v_2, \theta^p v_3, \theta^p v_4), \theta^{2p}\tau\right) = 1 \quad \forall v_1, v_2, v_3, v_4 \in M,\ \tau > 0, \tag{3.1}$$

and fulfilling the inequality

$$H(D\phi(v_1, v_2, v_3, v_4), \tau) \geq H^{'}(\Upsilon(v_1, v_2, v_3, v_4), \tau) \quad \forall v_1, v_2, v_3, v_4 \in M,\ \tau > 0. \tag{3.2}$$

If there exist $L = L[l]$ such that $v \to \gamma(v) = \Upsilon\left(\frac{v}{5}, \frac{v}{5}, \frac{v}{5}, \frac{v}{5}\right)$ has the property

$$H^{'}\left(L\frac{1}{\theta_l^2}\gamma(\theta_l v), \tau\right) = H^{'}\left(\gamma(v), \tau\right) \quad \forall v \in M,\ \tau > 0, \tag{3.3}$$

then there exist unique quadratic function $Q_2 : M \to N$ fulfilling the functional equation (1.2) and

$$H(\phi(v) - Q_2(v), \tau) \geq H^{'}\left(\frac{L^{1-l}}{1-L}\gamma(v), \tau\right) \quad \forall v \in M,\ \tau > 0.$$

Proof. Let d be a general metric on χ such that

$$d(t, s) = inf\left\{p \in (0, \infty)|H(t(v) - s(v), \tau) \geq H^{'}(\gamma(v), p\tau), v \in M, \tau > 0\right\}.$$

It is easy to see that (χ, d) is complete. Define $T : \chi \to \chi$ by $Tt(v) = \frac{1}{\theta_l^2}t(\theta_l v) \quad \forall v \in M$, for $t, s \in \chi$, we obtain

$$\begin{aligned} d(t, s) = p &\Rightarrow H\left(t(v) - s(v), \tau\right) \geq H^{'}(\gamma(v), p\tau) \\ &\Rightarrow H\left(\frac{t(\theta_l v)}{\theta_l^2} - \frac{s(\theta_l v)}{\theta_l^2}, \tau\right) \geq H^{'}(\gamma(\theta_l v), p\theta_l^2\tau) \\ &\Rightarrow H(Tt(v) - Ts(v), \tau) \geq H^{'}(\gamma(\theta_l v), p\theta_l^2\tau) \end{aligned} \tag{3.4}$$

$$\Rightarrow H(Tt(v) - Ts(v), \tau) \geq H^{'}(\gamma(v), pL\tau)$$
$$\Rightarrow d(Tt(v), Ts(v)) \geq pL$$
$$\Rightarrow d(Tt, Ts, \tau) \geq Ld(t, s) \quad \forall\, t, s \in \chi.$$

So, T is strictly contractive mapping on χ with Lipschitz constant L, switching (v_1, v_2, v_3, v_4) by (v, v, v, v) in (3.2), we have

$$H\left(\phi(5v) - 25\phi(v), \tau\right) \geq H^{'}\left(\Upsilon(v, v, v, v), \tau\right) \quad \forall v \in M,\ \tau > 0. \tag{3.5}$$

Using (N_3) in (3.5), we reach

$$H\left(\frac{\phi(5v)}{5^2} - \phi(v), \tau\right) \geq H^{'}\left(\frac{\Upsilon(v, v, v, v)}{5^2}, \tau\right) \quad \forall v \in M,\ \tau > 0, \tag{3.6}$$

with the help of (3.3) when $l = 0$, it follows from (3.6) that

$$\Rightarrow H\left(\frac{\phi(5v)}{5^2} - \phi(v), \tau\right) \geq H^{'}(L\gamma(v), \tau)$$
$$\Rightarrow d(T\phi, \phi) \leq L = L^1 = L^{1-l}. \tag{3.7}$$

Replacing v by $\frac{v}{5}$ in (3.5), we get

$$H\left(\phi(v) - 25\phi\left(\frac{v}{5}\right), \tau\right) \geq H^{'}\left(\Upsilon\left(\frac{v}{5}, \frac{v}{5}, \frac{v}{5}, \frac{v}{5}\right), \tau\right) \quad \forall v \in M,\ \tau > 0,$$

when $l = 1$, it follows from (3.7), we reach

$$\Rightarrow H\left(\phi(v) - 25\phi\left(\frac{v}{5}\right), \tau\right) \geq H^{'}(\gamma(v), \tau)$$
$$\Rightarrow d(\phi, T\phi) \leq 1 = L^0 = L^{1-l}. \tag{3.8}$$

Then from (3.7) and (3.8), we can conclude

$$\Rightarrow d(\phi, T\phi) \leq L^{1-l} < \infty.$$

Now, from Theorem 1.1 in both cases, it follows that there exists a fixed point Q_2 of T in χ such that

$$Q_2(v) = H - \lim_{p \to \infty} \frac{\phi(\theta^p v)}{\theta^{2p}} \quad \forall\, v \in M \text{ and } \tau > 0.$$

Replacing $(v_1, v_2, v_{3,}, v_4)$ by $(\theta_l^p v_1, \theta_l^p v_2, \theta_l^p v_3, \theta_l^p v_4)$ in (3.2), we arrive

$$H\left(\frac{1}{\theta_l^{2p}} D\phi(\theta_l^p v_1, \theta_l^p v_2, \theta_l^p v_3, \theta_l^p v_4), \tau\right) \geq H^{'}\left(\Upsilon(\theta_l^p v_1, \theta_l^p v_2, \theta_l^p v_3, \theta_l^p v_4), \theta_l^{2p}\tau\right)$$

for all $\tau > 0$ and all $v_1, v_2, v_3, v_4 \in M$. By similar procedure of Theorem 2.1, we show that the function $Q_2 : M \to N$ is quadratic and it fulfils (1.2). By Theorem 1.1, as Q_2 is unique fixed point of T in the set

$$\Delta = \left\{\phi \in \chi / d(\phi, Q_2) < \infty\right\}.$$

So that, Q_2 is a unique function such that

$$H(\phi(v) - Q_2(v), \tau) \geq H^{'}(\gamma(v), p\tau) \quad \forall v \in M,\ \tau > 0.$$

Again utilizing Theorem 1.1, we reach

$$d(\phi, Q_2) \leq \frac{1}{1 - L} d(\phi, T\phi)$$
$$\Rightarrow d(\phi, Q_2) \leq \frac{L^{1-l}}{1 - L}$$

$$\Rightarrow H(\phi(v) - Q_2(v), \tau) \geq H^{'}\left(\gamma(v)\frac{L^{1-l}}{1-L}, \tau\right) \quad \forall v \in M,\ \tau > 0.$$

□

Corollary 3.2. *Suppose a function $\phi : M \to N$ fulfils the inequality*

$$H(D\phi(v_1, v_2, v_3, v_4), \tau) \geq \begin{cases} H'(\eta, \tau), \\ H'\left(\eta||v_1||^a + ||v_2||^a + ||v_3||^a + ||v_4||^a, \tau\right), \\ H'\Big(\eta\Big(||v_1||^a + ||v_2||^a + ||v_3||^a + ||v_4||^a + \\ \qquad ||v_1||^{4a} \cdot ||v_2||^{4a} \cdot ||v_3||^{4a} \cdot ||v_4||^{4a}\Big), \tau\Big), \end{cases}$$

for all $v_1, v_2, v_3, v_4 \in M$ and $\tau > 0$, where η, a are constants with $\eta > 0$. Then there exists a unique quadratic mapping $Q_2 : M \to N$ such that

$$H(\phi(v) - Q_2(v), \tau) \geq \begin{cases} H^{'}\left(\eta, \frac{\tau}{|24|}\right), \\ H^{'}\left(4\eta||v||^a, \tau \mid 5^2 - 5^a \mid\right); & a \neq 2, \\ H^{'}\left(5\eta||v||^{4a}, \tau \mid 5^2 - 5^{4a} \mid\right); & a \neq \frac{2}{4}. \end{cases}$$

for all $v \in M$ and $\tau > 0$.

Proof. Setting

$$\Upsilon(v_1, v_2, v_3, v_4) \leq \begin{cases} \eta, \\ \eta(\sum_{m=1}^4 ||v_m||^a), \\ \eta(\prod_{m=1}^4 ||v_m||^a + \sum_{m=1}^4 ||v_m||^{4a}), \end{cases}$$

for all $v_1, v_2, v_3, v_4 \in M$. Then

$$H^{'}\left(\Upsilon\left(\theta_l^p v_1, \theta_l^p v_2, \theta_l^p v_3, \theta_l^p v_4\right), \theta_l^{2p}\tau\right) = \begin{cases} H^{'}(\eta, \theta_l^{2p}\tau), \\ H^{'}\left(\eta\sum_{m=1}^4 ||v_m||^a, \theta_l^{(2-a)p}\tau\right), \\ H^{'}\left(\eta(\sum_{m=1}^4 ||v_l||^{4a} + \prod_{m=1}^4 ||v_m||^a), \theta_m^{(2-4a)p}\tau\right). \end{cases}$$

$$= \begin{cases} \to \quad 1 \quad as \quad p \to \infty, \\ \to \quad 1 \quad as \quad p \to \infty, \\ \to \quad 1 \quad as \quad p \to \infty. \end{cases}$$

Thus, (2.1) is holds. But we get

$$\gamma(v) = \Upsilon\left(\frac{v}{5}, \frac{v}{5}, \frac{v}{5}, \frac{v}{5}\right)$$

has the property

$$H^{'}\left(L\frac{1}{\theta_l^2}\gamma(\theta_l v), \tau\right) \geq H^{'}(\gamma(v), \tau) \quad \forall v \in M,\ \tau > 0.$$

Hence,

$$H^{'}(\gamma(v), \tau) = H^{'}\left(\Upsilon\left(\frac{v}{5}, \frac{v}{5}, \frac{v}{5}, \frac{v}{5}\right), \tau\right)$$

$$= \begin{cases} H^{'}(\eta, \tau), \\ H^{'}\left(\frac{4}{5^a}\eta||v||^a, \tau\right), \\ H^{'}\left(\frac{5}{5^{4a}}\eta||v||^{4a}, \tau\right). \end{cases}$$

Now,

$$H^{'}\left(\frac{1}{\theta_l^2}\gamma(\theta_l v),\tau\right)=\begin{cases}H^{'}\left(\frac{\eta}{\theta_l^2},\tau\right),\\ H^{'}\left(\frac{\eta}{\theta_l^2}\left(\frac{4}{5^a}\right)||\theta_l v||^a,\tau\right),\\ H^{'}\left(\frac{\eta}{\theta_l^2}\left(\frac{5}{5^{4a}}\right)||\theta_l v||^{4a},\tau\right).\end{cases}$$

$$=\begin{cases}H^{'}(\theta_l^{-2}\gamma(v),\tau),\\ H^{'}(\theta_l^{a-2}\gamma(v),\tau),\\ H^{'}(\theta_l^{4a-2}\gamma(v),\tau).\end{cases}$$

Next, from the following cases for the conditions (i) and (ii).

Case(i): $L=5^{-2}\quad for\quad a=0\quad if\quad l=0.$

$$H(\phi(v)-Q_2(v),\tau)\geq H^{'}\left(\tfrac{L^{1-l}}{1-L}\gamma(v),\tau\right)\geq H^{'}\left(\tfrac{5^{-2}}{1-5^{-2}}\eta,\tau\right)\geq H^{'}(\eta,24\tau).$$

Case(ii): $L=\left(\frac{1}{5}\right)^{-2}\quad for\quad a=0\quad if\quad l=1.$

$$H(\phi(v)-Q_2(v),\tau)\geq H^{'}\left(\tfrac{L^{1-l}}{1-L}\gamma(v),\tau\right)\geq H^{'}\left(\frac{1}{1-\left(\frac{1}{5}\right)^{-2}}\eta,\tau\right)\geq H^{'}(\eta,-24\tau).$$

Case(iii): $L=(5)^{a-2}\quad for\quad a<2\quad if\quad l=0.$

$$H(\phi(v)-Q_2(v),\tau)\geq H^{'}\left(\frac{L^{1-l}}{1-L}\gamma(v),\tau\right)\geq H^{'}\left(\frac{5^{a-2}}{1-5^{a-2}}\frac{4\eta||v||^a}{5^a},\tau\right)$$
$$\geq H^{'}\left(4\eta||v||^a,\tau(5^2-5^a)\right).$$

Case(iv): $L=(5)^{2-a}\quad for\quad a>2\quad if\quad l=1.$

$$H(\phi(v)-Q_2(v),\tau)\geq H^{'}\left(\frac{L^{1-l}}{1-L}\gamma(v),\tau\right)\geq H^{'}\left(\frac{5^{2-a}}{1-5^{2-a}}\frac{4\eta||v||^a}{5^a},\tau\right)$$
$$\geq H^{'}\left(4\eta||v||^a,\tau(5^a-5^2)\right).$$

Case(v): $L=(5)^{4a-2}\quad for\quad a<\frac{1}{4}\quad if\quad l=0\,.$

$$H(\phi(v)-Q_2(v),\tau)\geq H^{'}\left(\frac{L^{1-l}}{1-L}\gamma(v),\tau\right)\geq H^{'}\left(\frac{5^{4a-2}}{1-5^{4a-2}}\frac{5\eta||v||^{4a}}{5^{4a}},\tau\right)$$
$$\geq H^{'}\left(5\eta||v||^{4a},\tau(5^2-5^{4a})\right).$$

Case(vi): $L=(5)^{2-4a}\quad for\quad a>\frac{1}{4}\quad if\quad l=1.$

$$H(\phi(v)-Q_2(v),\tau)\geq H^{'}\left(\frac{L^{1-l}}{1-L}\gamma(v),\tau\right)\geq H^{'}\left(\frac{5^{2-4a}}{1-5^{2-4a}}\frac{5\eta||v||^{4a}}{5^{4a}},\tau\right)$$
$$\geq H^{'}\left(5\eta||v||^{4a},\tau(5^{4a}-5^2)\right).$$

□

4. CONCLUSION

In this work, we investigated Hyers-Ulam stability results for the quadratic functional equation (1.2). In section 2, we examined Hyers-Ulam stability results of the quadratic functional equation (1.2) in fuzzy normed space by utilizing direct method. In section 3, we obtained Hyers-Ulam stability results of the quadratic functional equation (1.2) in fuzzy normed space by utilizing fixed point method.

REFERENCES

[1] Abdulaziz M. Alanazi, G. Muhiuddin, K. Tamilvanan, Ebtehaj N. Alenze, Abdelhalim Ebaid, K. Loganathan. Fuzzy stability results of finite variable additive functional equation: direct and fixed point methods, Mathematics, 8(2020), 1050; doi:10.3390/math8071050.

[2] M. Adam. On the stability of some quadratic functional equation, J. Nonlinear Sci. Appl., 4(1)(2011), p. 50–59.

[3] C. Baak. Cauchy-Rassias stability of Cauchy-Jensen additive mappings in Banach spaces, Acta Math. Sin. (Engl. Ser.) 22(6) (2006), p. 1789–1796.

[4] J. A. Baker. On a functional equation of Aczel and Chung, Aequationes Math. 46 (1993), p. 99–111.

[5] J. Chung. Stability of a generalized quadratic functional equation in Schwartz distributions, Acta Math. Sin. (Engl. Ser.) 25 (2009), p. 1459–1468.

[6] G. Z. Eskandani, A. R. Zamani, and H. Vaezi. Fuzzy approximation of an additive functional equation, J. Funct. Spaces Appl. 9(2) (2011), p. 205–215.

[7] I. M. Gelfand, and G. E. Shilov. Generalized Functions II, Academic Press, New York (1968).

[8] M. E. Gordji, M. Kamyar, and Th. M. Rassias. General cubic- quartic functional equation, Abs. Appl. Anal. 2011, Art. ID 463164, 18 pages.

[9] S. S. Jin, and Y. H. Lee. On the stability of the quadratic-additive type functional equation in random normed spaces via fixed point method, Korean J. Math. 20(1) (2012), p.19–31.

[10] K. W. Jun, and H. M. Kim. On the Hyers-Ulam-Rassias stability of a general cubic functional equation, Math. Inequal Appl. 6(2) (2003), p. 289–302.

[11] Y. H. Kim. On the Hyers-Ulam-Rassias stability of an equation of Davison, Indian J. Pure Appl. Math. 33(5) (2002), p. 713–726.

[12] C. Park, K. Tamilvanan, G. Balasubramanian, B. Noori, and A. Najati. On a functional equation that has the quadratic-multiplicative property, Open Mathematics, 18(2020), p. 837–845.

[13] C. Park, K. Tamilvanan, B. Noori, M. B. Moghimi, and A. Najati. Fuzzy normed spaces and stability of a generalized quadratic functional equation, AIMS Mathematics, 5(6)(2020), p. 7161-7174.

[14] J. M. Rassias. Solution of a stability problem of Ulam, Discuss. Math. 12 (1992), p. 95–103.

[15] J. M. Rassias. On the stability of the Euler-Lagrange functional equation, Chinese J. Math. 20(1992), p. 185-190.

[16] K. Tamilvanan, A. Nandhakumar, P. Muthukumar. Solution and stability of finite dimensional quadratic functional equation in Banach spaces, AIP Conference Proceedings 2261, 030057 (2020); https://doi.org/10.1063/5.0016871.

[17] K. Tamilvanan, J. R. Lee, and C. Park. Hyers-Ulam stability of a finite variable mixed type quadratic-additive functional equation in quasi-Banach spaces, AIMS Math., 5(6)(2020), p. 5993-6005.

[18] K. Tamilvanan, G. Balasubramanian, A. Charles Sagayaraj. Finite dimensional even-quadratic functional equation and its Ulam-Hyers stability, AIP Conference Proceedings 2261, 030002 (2020); https://doi.org/10.1063/5.0016865.

[19] A. Wiwatwanich, and P. Nakmahachalasint. On the stability of a cubic functional equation, Thai J. Math. (2008) p. 69–76.

K. TAMILVANAN

GOVERNMENT ARTS COLLEGE FOR MEN, KRISHNAGIRI-635 001, TAMIL NADU, INDIA.

Email address: tamiltamilk7@gmail.com

G. BALASUBRAMANIAN

GOVERNMENT ARTS COLLEGE FOR MEN, KRISHNAGIRI-635 001, TAMIL NADU, INDIA.

Email address: gbs_geetha@yahoo.com

K. LOGANATHAN

DEPARTMENT OF MATHEMATICS, ERODE ARTS & SCIENCE COLLEGE, ERODE - 638 009, TAMIL NADU, INDIA.

Email address: loganathankaruppusamy304@gmail.com

ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 3, Number 4 (2020), 261-272
ISSN: 2582-0818

ON PYTHAGOREAN FUZZY IDEAL OF SUBTRACTION SEMIGROUP AND NEAR SUBTRACTION SEMIGROUP

V. CHINNADURAI AND A. ARULSELVAM*

ABSTRACT. In this paper, we define the notions of Pythagorean fuzzy ideal of subtraction semigroup and near subtraction semigroup. Also, we discuss some of its properties with examples.

1. INTRODUCTION

In 1969, Abbott introduced the notion of subtraction algebra. He considered the system of the form $(\phi, \circ, \backslash)$ where ϕ is a set of functions closed under the composition $\circ$ of function where $(\phi, \circ)$ is a functional semigroup and $(\phi, \backslash)$ is a subtraction algebra [1]. Using this concept Schein[9] introduced the concept of subtraction semigroups in 1992. He proved that every subtraction semigroup is isomorphism to a different semigroup of invertible function. Zelink[13] studied an extraordinary kind of subtraction algebra called atomic subtraction algebra. A near subtraction semigroup satisfies all axioms of subtraction semigroup except one of the two distributive laws. Prince williams[8] defined the fuzzy ideal in near subtraction semigroup(NSS). Jun et al.[5] introduced the concept of ideals in subtraction algebra and gave some characterizations. Dheena et al.[3, 4] discussed and derived some properties of NSS, a generalization of subtraction semigroup. The concept of fuzzy set was initiated by Zadeh[12]. Mahalakshmi et al.[7] studied the notion of bi ideals of NSS. Chinnadurai[2] defined the concept of fuzzy ideals in algebraic structures. Kim et al.[6], studied some properties of intuitionistic fuzzy ideals of semigroups. Yager[10, 11] introduced the Pythagorean fuzzy set(PFS).
In this paper, we introduce the notions of Pythagorean fuzzy ideal in subtraction semigroup and near subtraction semigroup. Also we characterize the subtraction and near subtraction semigroup through fuzzification.

2. PRELIMINARIES

Now, we recall some new concepts of Pythagorean fuzzy ideals in NSS from the literature, which are required in the sequel.

2010 *Mathematics Subject Classification.* 00X00, 00G00, 00D00.
Key words and phrases. Pythagorean; Fuzzy set; subtraction; Near-subtraction; semigroup.
Received: November 12, 2020. Accepted: December 12, 2020. Published: December 31, 2020.
*Corresponding author.

The basic concepts of subtraction semigroup(SS), power set(PS), NSS [6] and zero symmetric are referred respectively.

Definition 2.1. A non-empty set X together with a binary operation $" - "$ is said to be a SA(subtraction algebra), if
(i) $j-(k-j)=j$
(ii) $j-(j-k)=k-(k-j)$
(iii) $(j-k)-l=(j-l)-k \ \forall j,k,l \in X$.

Definition 2.2. A subset I of SA X is called subalgebra of X if $j-k \in I \, \forall j,k \in X$. In SA the following holds:
(1) $j-0=j$ and $0-j=0$
(2) $j-(j-k) \leq k$
(3) $j \leq k$ if and only if $j=k-m$ for some $m \in X$
(4) $j \leq k$ implies $j-l \leq k-l$ and $l-k \leq l-j \, \forall l \in X$
(5) $j-(j-(j-k))=j-k$
(6) $(j-k)-j=0$
(7) $(j-k)-k=j-k$.

3. Pythagorean fuzzy ideal in subtraction semigroup

In this section X denotes subtraction semigroup(SS).

Definition 3.1. A Pythagorean fuzzy set(PFS) $D=(\pi,\vartheta)$ of X is called PFSS(Pythagorean fuzzy subtraction sub-semigroup) of X if
(1) $\pi(j-k) \geq \bigwedge\{\pi(j),\pi(k)\}$
(2) $\vartheta(j-k) \leq \bigvee\{\vartheta(j),\vartheta(k)\} \ \forall j,k \in X$.

Example 3.2. Let $X=\{0,j,k,l\}$ be a subtraction sub-semigroup with two binary operations $'-'$ and $'\cdot'$ is defined as follows.

$-$	0	j	k	l
0	0	0	0	0
j	j	0	j	0
k	k	k	0	0
l	l	k	j	0

$\cdot$	0	j	k	l
0	0	0	0	0
j	0	j	0	0
k	0	0	k	k
l	0	0	k	k

Define a Pythagorean fuzzy set $D=(\pi,\vartheta)$ where $\pi: X \to [0,1]$ by $\pi(0)=0.8, \pi(j)=0.6, \pi(k)=0.3, \pi(l)=0.2$. Then $\vartheta: X \to [0,1]$ by $\vartheta(0)=0.3, \vartheta(j)=0.4, \vartheta(k)=0.6, \vartheta(l)=0.7$ Then D is a PFSS of X. Hence $D=(\pi,\vartheta)$ is a subtraction sub-semigroup of X.

Definition 3.3. A PFS $D=(\pi,\vartheta)$ of X is called PFLI(resp. PFRI) of X, if $\forall j,k \in X$.
(1) $\pi(j) \geq \wedge\{\pi(j-k),\pi(k)\}$
(2) $\vartheta(j) \leq \vee\{\vartheta(j-k),\vartheta(k)\}$
(3) $\pi(jk) \geq \wedge\{\pi(j),\pi(k)\}$
(4) $\vartheta(jk) \leq \vee\{\vartheta(j),\vartheta(k)\}$
(5) $\pi(jk) \geq \pi(k)$ (resp.$\pi(jk) \geq \pi(j)$)
(6) $\vartheta(jk) \leq \vartheta(k)$ (resp.$\vartheta(jk) \leq \vartheta(j)$).
If π and ϑ are both PFLI and PFRI of X. Then π and ϑ are both PFI of X.

Example 3.4. Let $X = \{0, j, k, l\}$ be a SS with two binary operations $'-'$ and $'\cdot'$ is defined as follows.

$-$	0	j	k	l
0	0	0	0	0
j	j	0	j	0
k	k	k	0	0
l	l	k	j	0

$\cdot$	0	j	k	l
0	0	0	0	0
j	0	j	0	0
k	0	0	k	k
l	0	0	k	k

Define a PFS $D = (\pi, \vartheta)$ where $\pi : X \to [0, 1]$ by $\pi(0) = 0.8, \pi(j) = 0.6, \pi(k) = 0.3, \pi(l) = 0.2$. Then $\vartheta : X \to [0, 1]$ by $\vartheta(0) = 0.3, \vartheta(j) = 0.4, \vartheta(k) = 0.6, \vartheta(l) = 0.7$. Then D is a PFL (resp.PFR) ideals of X. Hence $D = (\pi, \vartheta)$ is a PFLI (PFRI) of X.

Definition 3.5. Let $D_1 = (\pi_{D_1}, \vartheta D_1)$ and $D_2 = (\pi_{D_2}, \vartheta_{D_2})$ be any two PFS of X. Then the following FSs of X are defined as follows.

$$(D_1 \star D_2)(j) = \begin{cases} (\pi_{D_1} \star \pi_{D_2})(j) = \begin{cases} \bigvee\limits_{j \le xy} \wedge\{\pi_D(x), \pi_D(y)\} & if j \le xy \\ [0,0] & otherwise. \end{cases} \\ (\vartheta_{D_1} \star \vartheta_{D_2})(j) = \begin{cases} \bigwedge\limits_{j \le xy} \vee\{\vartheta_D(x), \vartheta_D(y)\} & if j \le xy \\ [1,1] & otherwise. \end{cases} \end{cases}$$

$$(D_1 \cap D_2)(j) = \begin{cases} (\pi_{D_1} \cap \pi_{D_2})(j) \\ (\vartheta_{D_1} \cup \vartheta_{D_2})(j) \; \forall \, j \in X \end{cases}$$

$$(D_1 - D_2)(j) = \begin{cases} (\pi_{D_1} - \pi_{D_2})(j) = \begin{cases} \bigvee\limits_{j=k-l} \wedge\{\pi_D(k), \pi_D(l)\} & if j = k - l \; \forall \, j, k, l \in X \\ [0,0] & otherwise. \end{cases} \\ (\vartheta_{D_1} - \vartheta_{D_2})(j) = \begin{cases} \bigwedge\limits_{j=k-l} \vee\{\vartheta_D(k), \vartheta_D(l)\} & if j = k - l \; \forall \, j, k, l \in X \\ [1,1] & otherwise. \end{cases} \end{cases}$$

Theorem 3.1. *Every PFLI (resp. PFRI) of X is a PFSS of X.*

Proof. Let π and ϑ be an PFI of X. Then

$$\begin{aligned} \pi(j - k) &\ge \wedge\{\pi((j - k) - r), \pi(l)\} \; \forall \, l \in X \\ &\ge \wedge\{\pi((j - k) - l), \pi(j)\} \; for \; l = j \\ &= \wedge\{\pi(0), \pi(j)\} \\ &= \pi(j). \end{aligned}$$

Thus

$\pi(j - k) \ge \pi(j)$.

$$\begin{aligned} \vartheta(j - k) &\le \vee\{\vartheta((j - k) - l), \vartheta(r)\} \; \forall \, l \in X \\ &\le \vee\{\vartheta((j - k) - l), \vartheta(j)\} \; for \; l = j \\ &= \vee\{\vartheta(0), \vartheta(j)\} \\ &= \vartheta(j). \end{aligned}$$

Thus

$\vartheta(j - k) \le \vartheta(j)$.

Again consider

$\pi(j - k) \ge \wedge\{\pi((j - k) - l), \pi(l)\} \; \forall \, l \in X$

$$\begin{aligned} &\geq \wedge\{\pi((j-k)-k), \pi(k)\} \; for \; l = k \\ &= \wedge\{\pi(j-k), \pi(k)\} \text{ since } (j-k)-k = j-k \\ &= \wedge\{\pi(j), \pi(k)\}. \end{aligned}$$

And

$$\begin{aligned} \vartheta(j-k) &\leq \vee\{\vartheta((j-k)-l), \vartheta(l)\} \; \forall \; l \in X \\ &\leq \vee\{\vartheta((j-k)-k), \vartheta(k)\} \; for \; l = k \\ &= \vee\{\vartheta(j-k), \vartheta(k)\} \text{ since } (j-k)-k = j-k \\ &= \vee\{\vartheta(j), \vartheta(k)\}. \end{aligned}$$

Then π and ϑ are PFSS of X.
The converse is not true. □

Theorem 3.2. *If $D = (\pi, \vartheta)$ be an PFS of a SS X. Then*
(a)(i) $\pi \star \pi \leq \pi$
(ii) $\vartheta \star \vartheta \geq \vartheta$
(b)(i) $\pi(jk) \geq \wedge\{\pi(j), \pi(k)\}$
(ii) $\vartheta(jk) \leq \vee\{\vartheta(j), \vartheta(k)\} \; \forall \; j, k \in X$.

Proof. $(a) \Rightarrow (b)$ Let $x, y \in X$.
Consider

$$\begin{aligned} (\pi \star \pi)(jk) &= \bigvee_{xy \leq jk} \{\wedge\{\pi(j), \pi(k)\}\} \\ &\geq \wedge\{\pi(x), \pi(y)\}. \end{aligned}$$

By (a) $\pi \star \pi \leq \pi$.

$$\begin{aligned} \pi(xy) &\geq (\pi \star \pi)(xy) \\ &\geq \wedge\{\pi(x), \pi(y)\}. \end{aligned}$$

Hence $\pi(xy) \geq \wedge\{\pi(x), \pi(y)\}$.
And

$$\begin{aligned} (\vartheta \star \vartheta)(jk) &= \bigwedge_{xy \leq jk} \{\vee\{\vartheta(j), \vartheta(k)\}\} \\ &\leq \vee\{\vartheta(x), \vartheta(y)\}. \end{aligned}$$

By (a) $\vartheta \star \vartheta \geq \vartheta$.

$$\begin{aligned} \vartheta(xy) &\leq (\vartheta \star \vartheta)(xy) \\ &\leq \vee\{\vartheta(x), \vartheta(y)\}. \end{aligned}$$

Hence $\vartheta(xy) \leq \vee\{\vartheta(x), \vartheta(y)\}$.
$(b) \Leftrightarrow (a)$ Let $j \in X$.
Consider

$$\begin{aligned} (\pi \star \pi)(j) &= \bigvee_{j \leq xy} \wedge\{\pi(x), \pi(y)\} \\ &\leq \bigvee_{j \leq xy} \{\pi(xy)\} \\ &\leq \bigvee_{j \leq xy} \{\pi(j)\} \\ &= \pi(j). \end{aligned}$$

Thus $\pi \star \pi \leq \pi$. If j cannot be expressed as $j \leq xy$ then $(\pi \star \pi)(x) = 0 \leq \pi(j)$.
Thus $(\pi \star \pi)(j) \leq \pi(j) \; \forall j \in X$.
And
Let $j \in X$.
Consider

$$\begin{aligned} (\vartheta \star \vartheta)(j) &= \bigwedge_{j \leq xy} \vee\{\vartheta(x), \vartheta(y)\} \\ &\geq \bigwedge_{j \leq xy} \{\vartheta(xy)\} \end{aligned}$$

$$\geq \bigwedge_{j \leq xy} \{\vartheta(j)\}$$
$$= \vartheta(j).$$

Thus $\vartheta \star \vartheta \geq \vartheta$. If j cannot be expressed as $j \leq ab$ then $(\vartheta \star \vartheta)(x) = 0 \geq \vartheta(j)$.
Thus $(\vartheta \star \vartheta)(j) \geq \vartheta(j)\ \forall j \in X$.
This implies that $\pi \star \pi \leq$ and $\vartheta \star \vartheta \geq \vartheta$. □

Theorem 3.3. *Let $D =< \pi, \vartheta >$ be an PFS of X. If $D =< \pi, \vartheta >$ is an Pythagorean fuzzy(PF) sub-semigroup (PFLI, PFRI) of X. Then $\pi - \pi = \pi$ and $\vartheta - \vartheta = \vartheta$.*

Proof. Let $D =< \pi, \vartheta >$ be an PF sub-semigroup of X.
Let $p \in X$ then.

$$(\pi - \pi)(j) = \bigvee_{j=x-y} \{\wedge\{\pi(x), \pi(y)\}\}\ x, y \in X$$
$$\geq \wedge\{\pi(j), \pi(0)\} \text{ since } j = j - 0$$
$$= \pi(j).$$
$$(\vartheta - \vartheta)(j) = \bigwedge_{j=x-y} \{\vee\{\vartheta(x), \vartheta(y)\}\}\ x, y \in X$$
$$\leq \vee\{\vartheta(j), \vartheta(0)\} \text{ since } j = j - 0$$
$$= \vartheta(j).$$

On the other hand if $j = x - y$, $x, y \in X$, then

$$\pi(j) = \pi(x - y)$$
$$\geq \wedge\{\pi(x), \pi(y)\}$$
$$\geq \bigvee_{j=x-y} \{\wedge\{\pi(x), \pi(y)\}\}$$
$$= (\pi - \pi)(j).$$
$$\vartheta(j) = \vartheta(x - y)$$
$$\leq \vee\{\vartheta(x), \vartheta(y)\}$$
$$\leq \bigwedge_{j=x-y} \{\vee\{\vartheta(x), \vartheta(y)\}\}$$
$$= (\vartheta - \vartheta)(j).$$

Hence $\pi(j) = (\pi - \pi)(j)$ and $\vartheta(j) = (\vartheta - \vartheta)(j)\ \forall\ j \in X$.
Thus $\pi = \pi - \pi$ and $\vartheta = \vartheta - \vartheta$. □

Theorem 3.4. *Let D_1 and D_2 be any two PFS of X. If D_1 and D_2 are PFLI (resp. PFRI) of X. Then D_1 and D_2 is also PFLI (resp. PFRI) of X.*

Proof. Let D_1 and D_2 be any two PFLIs of X.
Let $j, k \in X$.
Consider

$$(\pi_1 \cap \pi_2)(jk) = \wedge\{\pi_1(jk), \pi_2(jk)\}$$
$$\geq \wedge\{\wedge\{\pi_1(j), \pi_1(k)\}, \wedge\{\pi_2(j), \pi_2(k)\}\}$$
$$\geq \wedge\{\wedge\{\pi_1(j), \pi_2(j)\}, \wedge\{\pi_1(k), \pi_2(k)\}\}$$
$$= \wedge\{(\pi_1 \cap \pi_2)(j), (\pi_1 \cap \pi_2)(k)\}.$$
$$(\pi_1 \cap \pi_2)(jk) = \wedge\{\pi_1(jk), \pi_2(jk)\}$$
$$\geq \wedge\{\pi_1(k), \pi_2(k)\}$$
$$= (\pi_1 \cap \pi_2)(k).$$
$$(\vartheta_1 \cup \vartheta_2)(jk) = \vee\{\vartheta_1(jk), \vartheta_2(jk)\}$$
$$\leq \vee\{\vee\{\vartheta_1(j), \vartheta_1(k)\}, \vee\{\vartheta_2(j), \vartheta_2(k)\}\}$$
$$\leq \vee\{\vee\{\vartheta_1(j), \vartheta_2(j)\}, \vee\{\vartheta_1(k), \vartheta_2(k)\}\Big\}$$
$$= \vee\{(\vartheta_1 \cup \vartheta_2)(j), (\vartheta_1 \cup \vartheta_2)(k)\}.$$
$$(\vartheta_1 \cup \vartheta_2)(jk) = \vee\{\vartheta_1(jk), \vartheta_2(jk)\}$$

$$\begin{aligned} &\leq \vee\{\vartheta_1(k), \vartheta_2(k)\} \\ &= (\vartheta_1 \cup \vartheta_2)(k). \end{aligned}$$

And

$$\begin{aligned} (\pi_1 \cap \pi_2)(j) &= \wedge\{\pi_1(j), \pi_2(j)\} \\ &\geq \wedge\{\bigwedge\{\pi_1(j-k), \pi_1(k)\}, \wedge\{\pi_2(j-k), \pi_2(k)\}\} \\ &\geq \wedge\{\wedge\{\pi_1(j-k), \pi_2(j-k)\}, \wedge\{\pi_1(k), \pi_2(k)\}\} \\ &= \wedge\{(\pi_1 \cap \pi_2)(j-k), (\pi_1 \cap \pi_2)(k)\}. \\ (\vartheta_1 \cup \vartheta_2)(j) &= \vee\{\vartheta_1(j), \vartheta_2(j)\} \\ &\leq \vee\{\vee\{\vartheta_1(j-k), \vartheta_1(k)\}, \vee\{\vartheta_2(j-k), \vartheta_2(k)\}\} \\ &\leq \vee\{\vee\{\vartheta_1(j-k), \vartheta_2(j-k)\}, \vee\{\vartheta_1(k), \vartheta_2(k)\}\} \\ &= \vee\{(\vartheta_1 \cup \vartheta_2)(j-k), (\vartheta_1 \cup \vartheta_2)(k)\}. \end{aligned}$$

This shows that the intersection of D_1 and D_2 two PFLI of X. □

Theorem 3.5. *If $D_i = (\pi_i, \vartheta_i | i \in \Omega)$ is a family of PFLI (resp. PFRI) of a SS X. Then $\bigcap_{i \in \Omega} D_i = (\cap_{i\in\Omega}\pi_i, \cup_{i\in\Omega}\vartheta_i)$ is also a PFLI (resp. PFRI) of X where Ω is any IS(index set).*

Proof. Let $D_i = (\pi_i, \vartheta_i | i \in \Omega)$ be a family of PFLI (resp. PFRI) of X.
Let $j, k \in X$ and $\pi(j) = \cap_{i\in\Omega}\pi_i(x) = \bigwedge \pi_i(j)$, $\vartheta(j) = \cup_{i\in\Omega}\vartheta_i(x) = \bigvee \vartheta_i(j)$.

$$\begin{aligned} \pi(j) &= \bigwedge \pi_i(j) \\ &\geq \bigwedge \wedge\{\pi_i(j-k), \pi_i(k)\} \\ &= \wedge\{\bigwedge \pi_i(j-k), \bigwedge \pi_i(k)\} \\ &= \wedge\{\cap\pi_i(j-k), \cap\pi_i(k)\} \\ &= \wedge\{\pi(j-k), \pi(k)\}. \\ \vartheta(j) &= \bigvee \vartheta_i(j) \\ &\leq \bigvee \vee\{\vartheta_i(j-k), \vartheta_i(k)\} \\ &= \vee\{\bigvee \vartheta_i(j-k), \bigvee \vartheta_i(k)\} \\ &= \vee\{\cup\vartheta_i(j-k), \cup\vartheta_i(k)\} \\ &= \vee\{\vartheta(j-k), \vartheta(k)\}. \\ \pi(jk) &= \bigwedge \pi_i(jk) \\ &\geq \bigwedge \wedge\{\pi_i(j), \pi_i(k)\} \\ &= \wedge\{\bigwedge \pi_i(j), \bigwedge \pi_i(k)\} \\ &= \wedge\{\cap\pi_i(j), \cap\pi_i(k)\} \\ &= \wedge\{\pi(j), \pi(k)\}. \\ \vartheta(jk) &= \bigvee \vartheta_i(jk) \\ &\leq \bigvee \vee\{\vartheta_i(j), \vartheta_i(k)\} \\ &= \vee\{\bigvee \vartheta_i(j), \bigvee \vartheta_i(k)\} \\ &= \vee\{\cup\vartheta_i(j), \cup\vartheta_i(k)\} \\ &= \vee\{\vartheta(j), \vartheta(k)\}. \end{aligned}$$

$$\pi(jk) = \bigwedge \pi_i(jk) \geq \bigwedge \pi_i(k) \geq \pi(k)$$

and

$$\vartheta(jk) = \bigvee \vartheta_i(jk) \leq \bigvee \vartheta_i(k) \leq \vartheta(k).$$

Hence, $\bigcap_{i \in \Omega} D_i = (\cap_{i\in\Omega}\pi_i, \cup_{i\in\Omega}\vartheta_i)$ is also a PFLI (resp. PFRI) of X. □

Theorem 3.6. *If $D = (\pi, \vartheta)$ is any PFS of a SS X then $D = (\pi, \vartheta)$ is a PFLI (resp. PFRI) of X if and only if every Pythagorean level set $\bigcup(D, t, n)$ is a LI (resp. RI) of X when it is non-empty.*

Proof. Suppose that $D = (\pi, \vartheta)$ is a PFLI (resp. PFRI) of X. Let $j, k, j-k \in \bigcup(D, t, n)$ $\forall\ t \in [0, 1]$ and $n \in [0, 1]$. Then $\pi(j) \geq t$, $\pi(j-k) \geq t$, $\pi(k) \geq t$ and $\vartheta(j) \leq n$,

$\vartheta(j-k) \leq n, \vartheta(k) \leq n$.
Suppose $k, j-k \in \bigcup(D,t,n)$ then $\pi(j) \geq \wedge\{\pi(j-k), \pi(k)\} \geq \wedge\{t,t\} = t$ and $\vartheta(j) \vee \{\vartheta(j-k), \vartheta(k)\} \leq \vee\{n,n\} = n$. Hence , $jk \in \bigcup(D,t,n)$.
Suppose $j, k \in \bigcup(D,t,n)$ then $\pi(jk) \geq \wedge\{\pi(j), \pi(k)\} \geq \wedge\{t,t\} = t$ and $\vartheta(jk) \vee \{\vartheta(j), \vartheta(k)\} \leq \vee\{n,n\} = n$. Hence , $jk \in \bigcup(D,t,n)$.
Let $j \in X$ and $k \in \bigcup(D,t,n)$ then $\pi(jk) \geq \pi(k) \geq t$ and $\vartheta(jk) \leq \vartheta \leq n$.
This implies that $jk \in \bigcup(D,t,n)$. Hence $\bigcup(D,t,n)$ is a LI of X.
Conversely, let $t \in [0,1]$ and $n \in [0,1]$ be $\ni \bigcup(D,t,n) \neq 0$ and $\bigcup(D,t,n)$ is a LI (RI) of X.
We assume that $\pi(j) \ngeq \wedge\{\pi(j-k), \pi(k)\}$ or $\vartheta(j) \nleq \vee\{\vartheta(j-k), \vartheta(k)\}$. If $\pi(j) \ngeq \wedge\{\pi(j-k), \pi(k)\}$ then $\exists\, t \in [0,1] \ni \pi(j) < t < \wedge\{\pi(j-k), \pi(k)\}$ hence $j-k, k \in \bigcup(D, t, \vee\{\vartheta(j-k), \vartheta(k)\})$ but $j \notin \bigcup(D, t, \vee\{\vartheta(j-k), \vartheta(k)\})$ which is contradiction.
If $\vartheta(j) \nleq \vee\{\vartheta(j-k), \vartheta(k)\}$ then $\exists\, n \in [0,1] \ni \vartheta(j) > n > \vee\{\vartheta(j-k), \vartheta(k)\}$ hence $j-k, k \in \bigcup(D, \wedge\{\pi(j-k), \pi(k)\}, n)$ but $j \notin \bigcup(D, \wedge\{\pi(j-k), \pi(k)\})$ which is contradiction. Hence, $\pi(j) \geq \wedge\{\pi(j-k), \pi(k)\}$ and $\vartheta(j) \leq \vee\{\vartheta(j-k), \vartheta(k)\}$.
Let us assume that $\pi(jk) \ngeq \wedge\{\pi(j), \pi(k)\}$ or $\vartheta(jk) \nleq \vee\{\vartheta(j), \vartheta(k)\}$. If $\pi(jk) \ngeq \wedge\{\pi(j), \pi(k)\}$ then $\exists\, t \in [0,1] \ni \pi(jk) < t < \wedge\{\pi(j), \pi(k)\}$
hence $j, k \in \bigcup(D, t, \vee\{\vartheta(j), \vartheta(k)\})$ but $jk \notin \bigcup(D, t, \vee\{\vartheta(j), \vartheta(k)\})$ which is contradiction.
If $\vartheta(jk) \nleq \vee\{\vartheta(j), \vartheta(k)\}$ then $\exists\, n \in [0,1] \ni \vartheta(jk) > n > \vee\{\vartheta(j), \vartheta(k)\}$ hence $j, k \in \bigcup(D, \wedge\{\pi(j), \pi(k)\}, n)$ but $jk \notin \bigcup(D, \wedge\{\pi(j), \pi(k)\})$ which is contradiction. Hence, $\pi(jk) \geq \wedge\{\pi(j), \pi(k)\}$ and $\vartheta(jk) \leq \vee\{\vartheta(j), \vartheta(k)\}$.
Assume that $\pi(jk) \ngeq \pi(k)$ or $\vartheta(jk) \nleq \vartheta(k)$. If $\pi(jk) \ngeq \pi(k)$ then $\exists\, t \in [0,1] \ni \pi(jk) < t < \pi(k)$ hence $k \in \bigcup(D, t, \vartheta(k))$ but $jk \notin \bigcup(D, t, \vartheta(k))$ which is contradiction. If $\vartheta(jk) \nleq \vartheta(k)$ then $\exists\, n \in [0,1] \ni \vartheta(jk) > n > \vartheta(k)$ hence $k \in \bigcup(D, \pi(k), n)$ but $jk \notin \bigcup(D, \pi(k), n)$ which is contradiction.
Hence, $\pi(jk) \geq \pi(k)$ and $\vartheta(jk) \leq \vartheta(k)$.
Therefore, $D = (\pi, \vartheta)$ is a PFLI (resp. PFRI) of X. $\square$

Theorem 3.7. *Let N be any non-empty subset of a SS X. Then N is a LI (resp. RI) of X if and only if the characteristic PFS $\chi_N = (\pi_{\chi_N}, \vartheta_{\chi_N})$ of N is X is a PFLI (resp. PFRI) of X.*

Proof. Assume that N is a LI of X. Let $j, k \in X$. Suppose that $\pi_{\chi_N}(j) < \wedge\{\pi_{\chi_N}(j-k), \pi_{\chi_N}(k)\}$ and $\vartheta_{\chi_N}(j) > \vee\{\vartheta_{\chi_N}(j-k), \vartheta_{\chi_N}(k)\}$. It follows that $\pi_{\chi_N}(j) = 0, \wedge\{\pi_{\chi_N}(j-k), \pi_{\chi_N}(k)\} = 1$ and $\vartheta_{\chi_N}(j) = 1, \vee\{\vartheta_{\chi_N}(j-k), \vartheta_{\chi_N}(k)\} = 0$. This implies that $j-k, k \in N$ but $j \notin N$, a contradicts to N being a SS of X.
Suppose that $\pi_{\chi_N}(jk) < \wedge\{\pi_{\chi_N}(j), \pi_{\chi_N}(k)\}$ and $\vartheta_{\chi_N}(jk) > \vee\{\vartheta_{\chi_N}(j), \vartheta_{\chi_N}(k)\}$. It follows that $\pi_{\chi_N}(jk) = 0, \wedge\{\pi_{\chi_N}(j), \pi_{\chi_N}(k)\} = 1$ and $\vartheta_{\chi_N}(jk) = 1$,
$\vee\{\vartheta_{\chi_N}(j), \vartheta_{\chi_N}(k)\} = 0$. This implies that $j, k \in N$ but $jk \notin N$, a contradicts to N.
Suppose that $\pi_{\chi_N}(jk) < \pi_{\chi_N}(k)$ and $\vartheta_{\chi_N}(jk) > \vartheta_{\chi_N}(k)$. It follows that $\pi_{\chi_N}(jk) = 0, \pi_{\chi_N}(k) = 1$ and $\vartheta_{\chi_N}(jk) = 1, \vartheta_{\chi_N}(k) = 0$. This implies that $k \in N$ but $jk \notin N$, a contradicts to N. This shows that $\chi_N = (\pi_{\chi_N}, \vartheta_{\chi_N})$ is a PFLI (resp. PFRI) of X.
Conversely, $\chi_N = (\pi_{\chi_N}, \vartheta_{\chi_N})$ is a PFLI(resp. PFRI) of X for any subset N of X.
Let $j-k, k \in N$ for any $j, k \in X$ then $\pi_{\chi_N}(j-k) = \pi_{\chi_N}(k) = 1$ and $\vartheta_{\chi_N}(j-k) = \vartheta_{\chi_N}(k) = 0$. Since χ_N is a PFLI (resp. PFRI) of X.
$\pi_{\chi_N}(j) \geq \wedge\{\pi_{\chi_N}(j-k), \pi_{\chi_N}(k)\} \geq \wedge\{1,1\} = 1$ and $\vartheta_{\chi_N}(j) \leq \vee\{\vartheta_{\chi_N}(j-k), \vartheta_{\chi_N}(k)\} \leq \wedge\{0,0\} = 0$. This implies that $j \in N$.
Let $j, k \in N$ for any $j, k \in X$ then $\pi_{\chi_N}(j) = \pi_{\chi_N}(k) = 1$ and $\vartheta_{\chi_N}(j) = \vartheta_{\chi_N}(k) = 0$.

Since χ_N is a PFLI (resp. PFRI) of X.
$\pi_{\chi_N}(jk) \geq \wedge\{\pi_{\chi_N}(j), \pi_{\chi_N}(k)\} \geq \wedge\{1,1\} = 1$ and $\vartheta_{\chi_N}(jk) \leq \vee\{\vartheta_{\chi_N}(j), \vartheta_{\chi_N}(k)\} \leq \wedge\{0,0\} = 0$. This implies that $jk \in N$.
Let $k \in N$ and $j \in X$ then $\pi_{\chi_N}(k) = 1$ and $\vartheta_{\chi_N}(k) = 0$. $\pi_{\chi_N}(jk) \geq \pi_{\chi_N}(k) \geq 1$ and $\vartheta_{\chi_N}(jk) \leq \vartheta_{\chi_N}(k) \geq 0$. This gives that $jk \in N$.
Hence, N is a LI (RI) of X. □

4. Pythagorean fuzzy ideal of near-subtraction semigroup

Definition 4.1. Let X be a near-subtraction semigroup(NSS), (X, D) be a PFI. A PFS $D = (\pi, \vartheta)$ is called a PFI of X, if
(1) $\pi(j-k) \geq \wedge\{\pi(j), \pi(k)\}$ and $\vartheta(j-k) \leq \vee\{\vartheta(j), \vartheta(k)\}$
(2) $\pi(xj - x(y-j)) \geq \pi(j)$ and $\vartheta(xj - x(y-j)) \leq \vartheta(j)$
(3) $\pi(jk) \geq \pi(j)$ and $\vartheta(jk) \leq \vartheta(j)$ $\forall j, k, l, x, y \in X$.
If $D = (\pi, \vartheta)$ is a PFLI of X if it satisfies (1),(2) and if $D = (\pi, \vartheta)$ is a PFRI of X if it satisfies (1) and (3).

Theorem 4.1. *If $D_i = (\pi_i, \vartheta_i | i \in \Omega)$ is a family of PFI of a NSS X then $\bigcap_{i\in\Omega} D_i = (\cap_{i\in\Omega}\pi_i, \cup_{i\in\Omega}\vartheta_i)$ is also a PFI of X where Ω is any IS(index set).*

Proof. If $\{D_i\}_{i\in\Omega}$ is a family of PFI of X.
Let $\cap\pi_i(x) = (\bigwedge \pi_i)(j) = \bigwedge \pi_i(j)$ and $\cup\vartheta_i(x) = (\bigvee \vartheta_i)(j) = \bigvee \vartheta_i(j)$. $\forall j, k \in X$, we have

$$\begin{aligned}(\cap_{i\in\Omega}\pi_i)(j-k) &= \textstyle\bigwedge\{\pi_i(j-k) | i \in \Omega\}\\ &\geq \textstyle\bigwedge \wedge\{\pi_i(j), \pi(k) | i \in \Omega\}\\ &= \wedge\{\textstyle\bigwedge\{\pi_i(j) | i \in \Omega\}, \bigwedge\{\pi_i(k) | i \in \Omega\}\}\\ &= \wedge\{\cap_{i\in\Omega}\pi_i(j), \cap_{i\in\Omega}\pi_i(k)\}.\end{aligned}$$

$$\begin{aligned}(\cup_{i\in\Omega}\vartheta_i)(j-k) &= \textstyle\bigvee\{\vartheta_i(j-k) | i \in \Omega\}\\ &\leq \textstyle\bigvee \vee\{\vartheta_i(j), \vartheta(k) | i \in \Omega\}\\ &= \vee\{\textstyle\bigvee\{\vartheta_i(j) | i \in \Omega\}, \bigvee\{\vartheta_i(k) | i \in \Omega\}\}\\ &= \vee\{\cup_{i\in\Omega}\vartheta_i(j), \cup_{i\in\Omega}\vartheta_i(k)\}.\end{aligned}$$

For all $j, x, y \in X$, we have

$$\begin{aligned}(\cap_{i\in\Omega}\pi_i)(xj - x(y-j)) &= \textstyle\bigwedge\{\pi_i(xj - x(y-j)) | i \in \Omega\}\\ &\geq \textstyle\bigwedge\{\pi_i(j) | i \in \Omega\}\\ &= \cap_{i\in\Omega}\pi_i(j).\end{aligned}$$

$$\begin{aligned}(\cup_{i\in\Omega}\vartheta_i)(xj - x(y-j)) &= \textstyle\bigvee\{\vartheta_i(xj - x(y-j)) | i \in \Omega\}\\ &\leq \textstyle\bigvee\{\vartheta_i(j) | i \in \Omega\}\\ &= \cup_{i\in\Omega}\vartheta_i(j).\end{aligned}$$

For all $j, k \in X$, we have

$$\begin{aligned}(\cap_{i\in\Omega}\pi_i)(jk) &= \textstyle\bigwedge\{\pi_i(jk) | i \in \Omega\}\\ &\geq \textstyle\bigwedge\{\pi_i(j) | i \in \Omega\}\\ &= \cap_{i\in\Omega}\pi_i(j).\end{aligned}$$

$$\begin{aligned}(\cup_{i\in\Omega}\vartheta_i)(jk) &= \textstyle\bigvee\{\vartheta_i(jk) | i \in \Omega\}\\ &\leq \textstyle\bigvee\{\vartheta_i(j) | i \in \Omega\}\\ &= \cup_{i\in\Omega}\vartheta_i(j).\end{aligned}$$

Hence, $\bigcap_{i\in\Omega} D_i = (\cap_{i\in\Omega}\pi_i, \cup_{i\in\Omega}\vartheta_i)$ is also a PFI of X. □

Definition 4.2. An PFS $D = (\pi, \vartheta)$ of X is said to be an PFBI of X if $\forall j, k \in X$
(i) $\pi(j-k) \geq \wedge\{\pi(j), \pi(k)\}$
(ii) $\vartheta(j-k) \leq \vee\{\vartheta(j), \vartheta(k)\}$
(iii) $(\pi \circ X \circ \pi) \cap (\pi \circ X) \star \pi \subseteq \pi$

(iv) $(\vartheta \circ X \circ \vartheta) \cup (\vartheta \circ X) \star \vartheta \supseteq \vartheta$.

Example 4.3. Let $X = \{0, j, k, l\}$ be a NSS with two binary operations $'-'$ and $'\cdot'$ is defined as follows.

$-$	0	j	k	l	$\cdot$	0	j	k	l
0	0	0	0	0	0	0	0	0	0
j	j	0	j	j	j	j	j	j	j
k	k	k	0	k	k	0	0	0	k
l	l	l	l	0	l	0	0	0	l

Define a PFS $D = (\pi, \vartheta)$ where $\pi : X \to [0, 1]$ by $\pi(0) = 0.8, \pi(j) = 0.6, \pi(k) = 0.3, \pi(l) = 0.2$. $(\pi \circ X \circ \pi)(0) = 0.8, (\pi \circ X \circ \pi)(j) = 0.7, (\pi \circ X \circ \pi)(k) = 0.5, (\pi \circ X \circ \pi)(l) = 0.2$, $(\pi \circ X) \star \pi(0) = 0.7, (\pi \circ X) \star \pi(j) = 0.6, (\pi \circ X) \star \pi(k) = 0.3, (\pi \circ X) \star \pi(l) = 0.1$ Then $\vartheta : X \to [0, 1]$ by $\vartheta(0) = 0.3, \vartheta(j) = 0.4, \vartheta(k) = 0.6, \vartheta(l) = 0.7$ $(\pi \circ X \circ \pi)(0) = 0.4, (\pi \circ X \circ \pi)(j) = 0.5, (\pi \circ X \circ \pi)(k) = 0.6, (\pi \circ X \circ \pi)(l) = 0.8$, $(\pi \circ X) \star \pi(0) = 0.3, (\pi \circ X) \star \pi(j) = 0.6, (\pi \circ X) \star \pi(k) = 0.7 and (\pi \circ X) \star \pi(l) = 0.8$.

Proposition 4.2. *Let $D = (\pi, \vartheta)$ be a PFS of X. If $D = (\pi, \vartheta)$ is a PFLI of X then $D = (\pi, \vartheta)$ is a PFBI of X.*

Proof. Let $j' \in X$ be such that $j' = xyz = jl - j(k - l)$, where $x, y, z, j, k, l \in X$. Then

$$\begin{aligned}
((\pi X \pi) \cap (\pi X \star \pi))(j') &= \wedge\{((\pi X \pi)(j'), (\pi X \star \pi))(j')\} \\
&= \wedge\{\bigvee_{j'=xyz} \wedge\{\pi(x).\pi(y), \pi(z)\}\}, \bigvee_{j'=jl-j(k-l)} \wedge\{(\pi X)(j), \pi(l)\} \\
&= \wedge\{\bigvee\{\pi(x), \pi(z)\}, \bigvee\{(\pi X)(j), \pi(l)\}\}.
\end{aligned}$$

(Since $\pi X \subseteq X$ and π is a PFLI, then $\pi(jl - j(k - l)) \geq \pi(l)$)

$$\begin{aligned}
&\leq \wedge\{X(x), X(z), X(j), \pi(jl - j(k - l))\} \\
&= \wedge\{1, 1, 1, \pi(jl - j(k - l))\} \\
&= \pi(jl - j(k - l)) \\
&= \pi(j').
\end{aligned}$$

If j' is not expressible as $j' = xyz = jl - j(k - l)$ then $(\pi X \pi \cap \pi X \star \pi)(j') = 0 \leq \pi(j')$. Then $\pi X \pi \cap \pi X \star \pi \subseteq \pi$. Hence D is a PFBI of X.

And

$$\begin{aligned}
((\vartheta X \vartheta) \cup (\vartheta X \star \vartheta))(j') &= \vee\{((\vartheta X \vartheta)(j'), (\vartheta X \star \vartheta))(j')\} \\
&= \vee\{\bigwedge_{j'=xyz} \vee\{\vartheta(x), \vartheta(y), \vartheta(z)\}\}, \bigwedge_{j'=jl-j(k-l)} \vee\{(\vartheta X)(j), \vartheta(l)\} \\
&= \vee\{\bigwedge\{\vartheta(x), \vartheta(z)\}, \bigwedge\{(\vartheta X)(j), \vartheta(l)\}\}.
\end{aligned}$$

(Since $\vartheta X \supseteq X$ and ϑ is a PFLI, then $\vartheta(jl - j(k - l)) \leq \vartheta(l)$)

$$\begin{aligned}
&\geq \vee\{X(x), X(z), X(j), \vartheta(jl - j(k - l))\} \\
&= \vee\{0, 0, 0, \vartheta(jl - j(k - l))\} \\
&= \vartheta(jl - j(k - l)) \\
&= \vartheta(j').
\end{aligned}$$

If j' is not expressible as $j' = xyz = jl - j(k - l)$ then $(\vartheta X \vartheta \cup \vartheta X \star \vartheta)(j') = 0 \geq \vartheta(j')$. Then $\vartheta X \vartheta \cup \vartheta X \star \vartheta \supseteq \vartheta$. Hence D is a PFBI of X. $\square$

Proposition 4.3. *Let $D = (\pi, \vartheta)$ be a PFS of X. If $D = (\pi, \vartheta)$ is a PFRI of X then $D = (\pi, \vartheta)$ is a PFBI of X.*

Proof. Let $j^{'} \in X$ be $\ni$ $j^{'} = xy = jl - j(k-l),\ x = x_1x_2$, where x, x_1, x_2, y, j, k and l are in X. Consider,

$$\begin{aligned}((\pi X\pi) \cap (\pi X \star \pi))(j^{'}) &= \wedge\{(\pi X\pi)(j^{'}), (\pi X \star \pi)(j^{'})\} \\ &= \wedge\{\bigvee_{j^{'}=xy} \wedge\{(\pi X)(x), \pi(y)\}, (\pi X \star \pi)(jl - j(k-l))\} \\ &= \wedge\{\bigvee_{j^{'}=xy} \wedge\{\bigvee_{x=x_1x_2} \wedge\{\pi(x_1), X(x_2)\}, \pi(y)\}, (\pi X\star\pi)(jl-j(k-l))\} \\ &= \wedge\{\bigvee_{j^{'}=xy} \wedge\{\bigvee_{x=x_1x_2} \{\pi(a_1)\}, \pi(b)\}, (\pi X \star \pi)(jl - j(k-l))\} \\ &= \wedge\{\pi(x_1), \pi(y), (\pi X \star \pi)(jl - j(k-l))\},\end{aligned}$$

$(since\ D = (\pi, \vartheta) is\ a\ PFRI,\ we\ have\ \pi(xy) = \pi(x_1x_2y) = \pi(x_1(x_2y)) \geq \pi(x_1))$

$$\leq \wedge\{\pi(xy), 1, 1\} = \pi(xy) = \pi(j^{'}).$$

If $j^{'}$ is not expressible as $j^{'} = xyz = jl - j(k-l)$ then $(\pi X\pi \cap \pi X \star \pi)(j^{'}) = 0 \leq \pi(j^{'})$. Then $\pi X\pi \cap \pi X \star \pi \subseteq \pi$. Hence D is a PFBI of X.

And

$$\begin{aligned}((\vartheta X\vartheta) \cup (\vartheta X \star \vartheta))(j^{'}) &= \vee\{(\vartheta X\vartheta)(j^{'}), (\vartheta X \star \vartheta)(j^{'})\} \\ &= \vee\{\bigwedge_{j^{'}=xy} \vee\{(\vartheta X)(x), \vartheta(y)\}, (\vartheta X \star \vartheta)(jl - j(k-l))\} \\ &= \vee\{\bigwedge_{j^{'}=xy} \vee\{\bigwedge_{x=x_1x_2} \vee\{\vartheta(x_1), X(x_2)\}, \vartheta(y)\}, (\vartheta X\star\vartheta)(jl-j(k-l))\} \\ &= \vee\{\bigwedge_{j^{'}=xy} \vee\{\bigwedge_{x=x_1x_2} \{\vartheta(x_1)\}, \vartheta(y)\}, (\vartheta X \star \vartheta)(jl - j(k-l))\} \\ &= \vee\{\vartheta(x_1), \vartheta(y), (\vartheta X \star \vartheta)(jl - j(k-l))\},\end{aligned}$$

$(since\ D = (\pi, \vartheta) is\ a\ PFRI,\ we\ have\ \vartheta(xy) = \vartheta(x_1x_2y) = \vartheta(x_1(x_2y)) \leq \vartheta(x_1))$

$$\geq \vee\{\vartheta(xy), 1, 1\} = \vartheta(xy) = \vartheta(j^{'}).$$

If $j^{'}$ is not expressible as $j^{'} = xyz = jl - j(k-l)$ then $(\vartheta X\vartheta \cup \vartheta X \star \vartheta)(j^{'}) = 0 \leq \vartheta(j^{'})$. Then $\vartheta X\vartheta \cup \vartheta X \star \vartheta \subseteq \vartheta$. Hence D is a PFBI of X. □

Theorem 4.4. *Let $D = (\pi, \vartheta)$ be a PFSA(Pythagorean fuzzy subalgebra) of X. If $DXD \subseteq D$, then D is a PFBI of X.*

Proof. Assume that π is a PFSA of X and $\pi X\pi \subseteq \pi$. Let $j \in X$. Then
$(\pi X\pi \cap \pi X \star \pi)(j) = \wedge\{(\pi X\pi)(j), (\pi X \star \pi)(j)\} \leq (\pi X\pi)(j) \leq \pi(j)$.
Thus $(\pi X\pi \cap \pi X \star \pi) \subseteq \pi$ and π is a PFBI of X and
assume that ϑ is a PFSA of X and $\vartheta X\vartheta \supseteq \vartheta$. Let $j \in X$. Then
$(\vartheta X\vartheta \cup \vartheta X \star \vartheta)(j) = \vee\{(\vartheta X\vartheta)(j), (\vartheta X \star \vartheta)(j)\} \geq (\vartheta X\vartheta)(j) \geq \vartheta(j)$.
Thus $(\vartheta X\vartheta \cup \vartheta X \star \vartheta) \supseteq \vartheta$ and ϑ is a PFBI of X. □

Theorem 4.5. *If X is a ZSNSS(zero symmetric near-subtraction semigroup) and D is a PFBI of X then $\pi X\pi \subseteq \pi$ and $\vartheta X\vartheta \supseteq \vartheta$.*

Proof. Let π be a PFBI of X. Then $\pi X\pi \cap \pi X \star \pi \subseteq \pi$. Clearly $\pi(0) \geq \pi(j)$. Thus $(\pi X)(0) \geq (\pi X)(j)\ \forall j \in X$. Since X is a ZSNSS, $\pi X\pi \subseteq \pi X \star \pi$. So $\pi X\pi \cap \pi X \star \pi = \pi X\pi \subseteq \pi$,
and let ϑ be a PFBI of X. Then $\vartheta X\vartheta \cup \vartheta X \star \vartheta \supseteq \vartheta$. Clearly $\vartheta(0) \leq \vartheta(j)$. Thus $(\vartheta X)(0) \leq (\vartheta X)(j)\ \forall j \in X$. Since X is a ZSNSS, $\vartheta X\vartheta \supseteq \vartheta X \star \vartheta$. So $\vartheta X\vartheta \cup \vartheta X \star \vartheta = \vartheta X\vartheta \supseteq \vartheta$. Which is the required result. □

Theorem 4.6. *Let $D = (\pi, \vartheta)$ be a PFBI of a ZSNSS X. Then $\pi(jkl) \geq \wedge\{\pi(j), \pi(l)\}$ and $\pi(jkl) \leq \vee\{\pi(j), \pi(l)\}$.*

Proof. Assume that π is a PFBI of ZSNSS X. BY Theorem 3.19 $\pi X \pi \subseteq \pi$. Let $j, k, l \in X$. Then

$$\begin{aligned}
\pi(jkl) &\geq (\pi X \pi)(jkl) \\
&= \bigvee_{jkl=xy} \wedge\{(\pi X)(x), \pi(y)\} \\
&\geq \wedge\{(\pi X)(jk), \pi(l)\} \\
&\geq \wedge\{(\pi X)(j), X(k), \pi(l)\} \\
&= \wedge\{(\pi X)(j), 1, \pi(l)\} \\
&= \wedge\{(\pi X)(j), \pi(l)\}.
\end{aligned}$$

Thus $\pi(pqr) \geq \wedge\{\pi(j), \pi(l)\}$.
And

$$\begin{aligned}
\vartheta(jkl) &\geq (\vartheta X \vartheta)(jkl) \\
&= \bigwedge_{jkl=xy} \vee\{(\vartheta X)(x), \vartheta(y)\} \\
&\leq \vee\{(\vartheta X)(jk), \vartheta(l)\} \\
&\leq \vee\{(\vartheta X)(j), X(k), \vartheta(l)\} \\
&= \vee\{(\vartheta X)(j), 1, \vartheta(l)\} \\
&= \vee\{(\vartheta X)(j), \vartheta(l)\}.
\end{aligned}$$

Thus $\vartheta(jkl) \leq \vee\{\vartheta(j), \vartheta(l)\}$. □

Theorem 4.7. *Let $D_1 = (\pi_1, \vartheta_1)$ and $D_2 = (\pi_2, \vartheta_2)$ be any two PFBI of X. Then $D_1 \cap D_2$ is also a PFBI of X.*

Proof. Let π_1 and π_2 be any two PFBI of X. Let $j, k \in X$. Then

$$\begin{aligned}
(\pi_1 \cap \pi_2)(j-k) &= \wedge\{\pi_1(j-k), \pi_2(j-k)\} \\
&\geq \wedge\{\wedge\{\pi_1(j), \pi_1(k)\}, \wedge\{\pi_2(j), \pi_2(k)\}\} \\
&= \wedge\{\wedge\{\pi_1(j), \pi_2(j)\}, \wedge\{\pi_1(k), \pi_2(k)\}\} \\
&= \wedge\{(\pi_1 \cap \pi_2)(j), (\pi_1 \cap \pi_2)(k)\}.
\end{aligned}$$

Let $j^{'} \in X$. Choose $x, y, j, k, l \in X$ such that $j^{'} = xyz = jl - j(k-l)$. Since π_1 and π_2 are PFBI of X, we have

$$\begin{aligned}
&= \wedge\{ \bigvee_{j^{'}=xyz} \wedge\{\pi_1(x), \pi_1(z)\}, \bigvee_{j^{'}=jl-j(k-l)} \pi_1\} \leq \pi_1(j) \\
&= \wedge\{ \bigvee_{j^{'}=xyz} \wedge\{\pi_2(x), \pi_2(z)\}, \bigvee_{j^{'}=jl-j(k-l)} \pi_2\} \leq \pi_2(j).
\end{aligned}$$

Now

$$\begin{aligned}
&= \wedge\{((\pi_1 \cap \pi_2)X(\pi_1 \cap \pi_2))(j^{'}), ((\pi_1 \cap \pi_2)X \star (\pi_1 \cap \pi_2))(j^{'})\} \\
&= \wedge\{ \bigvee_{j^{'}=xyz} \wedge\{(\pi_1 \cap \pi_1)(x), (\pi_1 \cap \pi_2)(z)\}, \bigvee_{j^{'}=jl-j(k-l)} (\pi_1 \cap \pi_2)(l)\} \\
&= \wedge\{ \bigvee_{j^{'}=xyz} \wedge\{\wedge\{\pi_1(x), \pi_2(x)\}, \wedge\{\pi_1(z), \pi_2(z)\}\}, \bigvee_{j^{'}=jl-j(k-l)} \wedge\{\pi_1(l), \pi_2(l)\}\} \\
&= \wedge\{\wedge\{ \bigvee_{j^{'}=xyz} \wedge\{\pi_1(x), \pi_1(z)\}, \bigvee_{j^{'}=jl-j(k-l)} \pi_1(l)\}, \wedge\{ \bigvee_{j^{'}=xyz} \wedge\{\pi_2(x), \pi_2(z)\}, \\
&\qquad \bigvee_{j^{'}=jl-j(k-l)} \pi_2(l)\}\} \\
&\leq \wedge\{\pi_1(j), \pi_2(j)\} \\
&= (\pi_1 \cap \pi_2)(j).
\end{aligned}$$

Thus $\pi_1 \cap \pi_2$ is a PFBI of X. and

$$\begin{aligned}
(\vartheta_1 \cup \vartheta_2)(j-k) &= \vee\{\vartheta_1(j-k), \vartheta_2(j-k)\} \\
&\leq \vee\{\vee\{\vartheta_1(j), \vartheta_1(k)\}, \vee\{\vartheta_2(j), \vartheta_2(k)\}\} \\
&= \vee\{\vee\{\vartheta_1(j), \vartheta_2(j)\}, \vee\{\vartheta_1(k), \vartheta_2(k)\}\} \\
&= \vee\{(\vartheta_1 \cup \vartheta_2)(j), (\vartheta_1 \cup \vartheta_2)(k)\}.
\end{aligned}$$

Let $j^{'} \in X$. Choose $x, y, j, k, l \in X, \ni\ j^{'} = xyz = jl - j(k-l)$. Since ϑ_1 and ϑ_2 are PFBI of X, we have

$$= \vee\{\bigwedge_{j^{'}=xyz} \vee\{\vartheta_1(x), \vartheta_1(z)\}, \bigwedge_{j^{'}=jl-j(k-l)} \vartheta_1\} \geq \vartheta_1(j)$$
$$= \vee\{\bigwedge_{j^{'}=xyz} \vee\{\vartheta_2(x), \vartheta_2(z)\}, \bigwedge_{j^{'}=jl-j(k-l)} \vartheta_2\} \geq \vartheta_2(j).$$

Now

$$\begin{aligned}
&= \vee\{((\vartheta_1 \cup \vartheta_2)X(\vartheta_1 \cup \vartheta_2))(j^{'}), ((\vartheta_1 \cup \vartheta_2)X \star (\vartheta_1 \cup \vartheta_2))(j^{'})\}\\
&= \vee\{\bigwedge_{j^{'}=xyz} \vee\{(\vartheta_1 \cup \vartheta_1)(x), (\vartheta_1 \cup \vartheta_2)(z)\}, \bigwedge_{j^{'}=jl-j(k-l)} (\vartheta_1 \cup \vartheta_2)(l)\}\\
&= \vee\{\bigwedge_{j^{'}=xyz} \vee\{\vee\{\vartheta_1(x), \vartheta_2(x)\}, \vee\{\vartheta_1(z), \vartheta_2(z)\}\}, \bigwedge_{j^{'}=jl-j(k-l)} \vee\{\vartheta_1(l), \vartheta_2(l)\}\}\\
&= \vee\{\vee\{\bigwedge_{j^{'}=xyz} \vee\{\vartheta_1(x), \vartheta_1(z)\}, \bigwedge_{j^{'}=jl-j(k-l)} \vartheta_1(l)\}, \vee\{\bigwedge_{j^{'}=xyz} \vee\{\vartheta_2(x), \vartheta_2(z)\},\\
&\quad \bigwedge_{j^{'}=jl-j(k-l)} \vartheta_2(l)\}\}\\
&\geq \vee\{\vartheta_1(j), \vartheta_2(j)\}\\
&= (\vartheta_1 \cup \vartheta_2)(j).
\end{aligned}$$

Thus $\vartheta_1 \cup \vartheta_2$ is a PFBI of X. $\square$

REFERENCES

[1] J. C. Abbott. Sets, Lattices and Boolean algebras, Allyn and Bacon, Boston, Year 1969.

[2] V. Chinnadurai. Fuzzy ideals in algebraic structures. Lap Lambert Academic Publishing, Year 2013.

[3] P. Dheena and G. Satheeshkumar. On strongly regular near-subtraction semigroups, Commun. Korean Math. Soc., Year 2007, p 323-330.

[4] P. Dheena and G. Satheeshkumar. Weakly prime left ideals in near-subtraction semigroups, Commun. Korean Math. Soc., 23(3), Year 2008, p 325-331.

[5] Y. B. Jun, Kyung Ja Lee and Asghar Khan. Ideal theory of subtraction algebras, Sci. Math. Jpn, 61, Year 2005, p 459-464.

[6] H.S. Kim and Y.B. Jun. Intuitionistic fuzzy ideals of semigroups, Indian J. Pure Appl. Math., 33(4), Year 2002, p 443-449.

[7] V. Mahalakshmi, S. Maharasi and S. Jayalakshmi. Bi-ideals of Near-Subtraction semigroup, Indian Advances in Algebra, 6(1), Year 2013, p 35-48.

[8] D. R. Prince Williams. Fuzzy ideals in near-subtraction semigroups, International Scholarly and Scientific Research and Innovation, 2(7), Year 2008, p 625-632.

[9] B.M. Schein. Difference semigroups, Algebra, 20, Year 1992, p 2513-2169.

[10] R.R. Yager. Pythagorean fuzzy subsets, In: Proc Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, Canada, Year 2013, p 57-61.

[11] R.R. Yager. Pythagorean membership grades in multicriteria decision making, IEEE Transaction on Fuzzy Systems, 22, Year 2014, p 958-965.

[12] L. A. Zadeh. Fuzzy sets, Information and Control, 8, Year 1965, p 338-353.

[13] B. Zelinka. Subtraction semigroups, Math. Bohemia, 120, Year 1995, p 445-447.

V. CHINNADURAI

DEPARTMENT OF MATHEMATICS, ANNAMALAI UNIVERSITY, ANNAMALAINAGAR-608002, TAMILNADU, INDIA.

Email address: kv.chinnadurai@yahoo.com

A. ARULSELVAM

DEPARTMENT OF MATHEMATICS, ANNAMALAI UNIVERSITY, ANNAMALAINAGAR-608002, TAMILNADU, INDIA.

Email address: arulselvam.a91@gmail.com

ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 3, Number 4 (2020), 273-284
ISSN: 2582-0818
© http://www.technoskypub.com

r-FUZZY $\Re s$-COMPACTNESS AND r-FUZZY $\Re s$-CONNECTEDNESS IN THE SENSE OF $\check{S}$OSTAK'S

E. ELAVARASAN

ABSTRACT. The purpose of this paper is to introduce the concepts of fuzzy regular semi compactness, fuzzy regular semi connectedness, fuzzy regular semi strongly connectedness and fuzzy regular semi-C_5-connectedness. Some interesting properties of these notions are studied. In this connection, interrelations are discussed. Examples are provided wherever necessary.

1. INTRODUCTION

$\check{S}$ostak [11], introduced the concept of fuzzy topological spaces as an extension of Chang's fuzzy topological spaces [2]. It has been developed in many direction [4, 7, 9]. Mashhour et. al., [8], A. M. Zahran [13] and E. E. Kerre et. al., [6] introduced the notion of fuzzy regular semi open and regular semi closed sets and investigate the relationship among fuzzy regular semi continuity and fuzzy regular semi irresolute mappings. Recently, Vadivel and Elavarasan [12] introduce and study the concept of fuzzy regular semi open sets and fuzzy regular semi continuous functions in fuzzy topological spaces in the sense of $\check{S}$ostak's. In this paper, we introduce the concepts of r-fuzzy regular semi compactness, r-fuzzy regular semi connectedness, r-fuzzy regular semi strongly connectedness and r-fuzzy regular semi-C_5-connectedness in the sense of $\check{S}$ostak's. Some interesting properties of these notions are studied. In this connection, interrelations are discussed. Examples are provided wherever necessary.

2. PRELIMINARIES

Throughout this paper, let X be a non-empty set, $I = [0,1]$, $I_0 = (0,1]$. A fuzzy set λ of X is a mapping $\lambda : X \to I$, and I^X be the family of all fuzzy sets on X. The complement of a fuzzy set λ is denoted by $\overline{1} - \lambda$. For $\lambda \in I^X$, $\overline{\lambda}(x) = \lambda$ for all $x \in X$. For each $x \in X$ and $t \in I_0$, a fuzzy point x_t is defined by $x_t(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$ Let $Pt(X)$

2010 *Mathematics Subject Classification.* 54A40, 03E72, 54D30.

Key words and phrases. r-fuzzy $\Re s$-compact, r-fuzzy $\Re s$-connected, r-fuzzy $\Re s$-strongly connected and r-fuzzy $\Re s$-C_5-connected.

Received: November 17, 2020. Accepted: December 14, 2020. Published: December 31, 2020.

be the family of all fuzzy points in X. All other notations and definitions are standard in the fuzzy set theory.

Definition 2.1. [11] A function $\tau : I^X \to I$ is called a fuzzy topology on X if it satisfies the following conditions:

(1) $\tau(\overline{0}) = \tau(\overline{1}) = 1$,
(2) $\tau(\bigvee_{i\in J} \mu_i) \geq \bigwedge_{i\in J} \tau(\mu_i)$, for any $\{\mu_i : i \in J\} \leq I^X$.
(3) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, for all $\mu_1,\ \mu_2 \in I^X$.

The pair (X, τ) is called a fuzzy topological space (for short, fts). A fuzzy set λ is called an r-fuzzy open (for short, r-fo) if $\tau(\lambda) \geq r$ and a fuzzy set λ is called an r-fuzzy closed (for short, r-fc) if $\tau(\overline{1} - \lambda) \geq r$.

Theorem 2.1. [3] *Let (X, τ) be a fts. Then for each $\lambda \in I^X$ and $r \in I_0$, we define an operator $C_\tau : I^X \times I_0 \to I^X$ as follows: $C_\tau(\lambda,\ r) = \bigwedge\{\mu \in I^X : \lambda \leq \mu,\ \tau(\overline{1}-\mu) \geq r\}$. For $\lambda,\ \mu \in I^X$ and $r, s \in I_0$, the operator C_τ satisfies the following statements:*

(C1) $C_\tau(\overline{0}, r) = \overline{0}$,
(C2) $\lambda \leq C_\tau(\lambda,\ r)$,
(C3) $C_\tau(\lambda,\ r) \vee C_\tau(\mu,\ r) = C_\tau(\lambda \vee \mu,\ r)$,
(C4) $C_\tau(\lambda,\ r) \leq C_\tau(\lambda,\ s)$ *if* $r \leq s$,
(C5) $C_\tau(C_\tau(\lambda,\ r),\ r) = C_\tau(\lambda,\ r)$.

Theorem 2.2. [3] *Let (X, τ) be a fts. Then for each $\lambda \in I^X$ and $r \in I_0$, we define an operator $I_\tau : I^X \times I_0 \to I^X$ as follows: $I_\tau(\lambda,\ r) = \bigvee\{\mu \in I^X : \mu \leq \lambda,\ \tau(\mu) \geq r\}$. For $\lambda,\ \mu \in I^X$ and $r, s \in I_0$, the operator I_τ satisfies the following statements:*

(I1) $I_\tau(\overline{1}, r) = \overline{1}$,
(I2) $I_\tau(\lambda,\ r) \leq \lambda$,
(I3) $I_\tau(\lambda,\ r) \wedge I_\tau(\mu,\ r) = I_\tau(\lambda \wedge \mu,\ r)$,
(I4) $I_\tau(\lambda,\ r) \leq I_\tau(\lambda,\ s)$ *if* $s \leq r$,
(I5) $I_\tau(I_\tau(\lambda,\ r),\ r) = I_\tau(\lambda,\ r)$.
(I6) $I_\tau(\overline{1} - \lambda, r) = \overline{1} - C_\tau(\lambda, r)$ *and* $C_\tau(\overline{1} - \lambda, r) = \overline{1} - I_\tau(\lambda,\ r)$

Definition 2.2. [10] Let $(X,\ \tau)$ be a fts, $\lambda \in I^X$ and $r \in I_0$. Then a fuzzy set λ is called

(1) r-fuzzy regular open (for short, r-fro) if $\lambda = I_\tau(C_\tau(\lambda, r), r)$.
(2) r-fuzzy regular closed (for short, r-frc) if $\lambda = C_\tau(I_\tau(\lambda, r), r)$.

Definition 2.3. [12] Let $(X,\ \tau)$ be a fts and $\lambda \in I^X,\ r \in I_0$. Then

(1) λ is called r-fuzzy regular semi open (for short, r-frso) if there exists r-fro set $\mu \in I^X$ and $\mu \leq \lambda \leq C_\tau(\mu, r)$.
(2) λ is called r-fuzzy regular semi closed (for short, r-frsc) if there exists r-frc set $\mu \in I^X$ and $I_\tau(\mu, r) \leq \lambda \leq \mu$.
(3) The r-fuzzy regular semi interior of λ, denoted by $RSI_\tau(\lambda, r)$, is defined by $RSI_\tau(\lambda, r) = \bigvee\{\mu \in I^X \mid \mu \leq \lambda,\ \mu \text{ is } r\text{-frso}\}$.
(4) The r-fuzzy regular semi closure of λ, denoted by $RSC_\tau(\lambda, r)$ is defined by $RSC_\tau(\lambda, r) = \bigwedge\{\mu \in I^X \mid \mu \geq \lambda,\ \mu \text{ is } r\text{-frsc}\}$.

We denote the set of all r-frso sets and r-frsc sets by $FRSO(X)$ and $FRSC(X)$.

Theorem 2.3. [12] *Let $(X,\ \tau)$ be a smooth topological space. For $\lambda \in I^X,\ r \in I_0$, the following statements are equivalent:*

(1) λ *is* r*-frso.*
(2) $\overline{1} - \lambda$ *is* r*-frso.*

(3) $I_\tau(\lambda, r) = I_\tau(C_\tau(\lambda, r), r)$.
(4) $C_\tau(\lambda, r) = C_\tau(I_\tau(\lambda, r), r)$.

Definition 2.4. [12] Let $(X,\ \tau)$ and $(Y,\ \eta)$ be fts's. Let $f : (X,\ \tau) \to (Y,\ \eta)$ be a mapping. Then f is said to be:

(1) fuzzy regular semi irresolute (resp. fuzzy regular semi continuous) iff $f^{-1}(\mu)$ is r-frso for each r-frso set $\mu \in I^Y$ (resp. $\mu \in I^Y,\ \eta(\mu) \geq r$).
(2) fuzzy regular semi irresolute open (resp. fuzzy regular semi open) iff $f(\lambda)$ is r-frso in Y for each r-frso set $\lambda \in I^X$ (resp. $\lambda \in I^X,\ \tau(\lambda) \geq r$).
(3) fuzzy regular semi irresolute closed (resp. fuzzy regular semi closed) iff $f(\lambda)$ is r-frsc in Y for each r-frsc set $\lambda \in I^X$ (resp. $\lambda \in I^X,\ \tau(\overline{1} - \lambda) \geq r$).

Definition 2.5. [1] Let $(X,\ \tau)$ and $(Y,\ \sigma)$ be a fts's. Let $f : (X,\ \tau) \to (Y,\ \sigma)$ be a function. Then f is called

(1) weakly continuous if for each $\mu \in I^Y$, where $\sigma(\mu) \geq r,\ r \in I_0,\ f^{-1}(\mu) \leq I_\tau(f^{-1}(C_\sigma(\mu, r)), r)$.
(2) weakly open if for each $\mu \in I^X$, where $\tau(\mu) \geq r,\ r \in I_0,\ f(\mu) \leq I_\sigma(f(C_\tau(\mu, r)), r)$.

Definition 2.6. [5] A fts (X, τ) is called an r-fuzzy compact (r-fuzzy nearly compact and r-fuzzy almost compact) if and only if for every family $\{\lambda_i | i \in J\}$ in $\{\lambda : \lambda \in I^X, \tau(\lambda) \geq r\}$ such that $\bigvee_{i\in J} \lambda_i = \overline{1}$, there exists a finite subset J_0 of J such that $\bigvee_{i\in J_0} \lambda_i = \overline{1}$ (resp. $\bigvee_{i\in J_0} I_\tau(C_\tau(\lambda_i, r), r) = \overline{1}$ and $\bigvee_{i\in J_0} C_\tau(\lambda_i, r) = \overline{1}$).

Theorem 2.4. *Let (X, τ) and (Y, σ) be two fts and $f : (X, \tau) \to (Y, \sigma)$ is fuzzy weakly open and fuzzy weakly continuous, then $f^{-1}(\lambda)$ is an r-fro (resp. r-frc) set for every r-fro $\lambda \in I^Y$, $r \in I_0$.*

Proof. Let λ be an r-fro set in Y, we have $\sigma(\lambda) \geq r$. Since f is fuzzy weakly continuous, $\tau(f^{-1}(\lambda) \geq r$. Hence $f^{-1}(\lambda) = I_\tau(f^{-1}(\lambda), r) \leq I_\tau(C_\tau(f^{-1}(\lambda), r), r)$. Since f is fuzzy weakly open, $f(I_\tau(C_\tau(f^{-1}(\lambda), r), r)) \leq I_\sigma(f(C_\tau(f^{-1}(\lambda), r), r))$. Since f is fuzzy weakly continuous, $I_\sigma(f(C_\tau(f^{-1}(\lambda), r), r)) \leq I_\sigma(ff^{-1}(C_\sigma(\lambda, r), r)) \leq I_\sigma(C_\sigma(\lambda, r), r)) = \lambda$. Hence $I_\tau(C_\tau(f^{-1}(\lambda), r), r) \leq f^{-1}(\lambda)$. Thus $f^1(\lambda)$ is r-fro. An r-frc case will be similar. □

3. r-FUZZY $\Re s$-COMPACTNESS

The most important of all covering properties is compactness. In this section, we introduce the concept of fuzzy $\Re s$-compactness and study some of its basic properties.

Definition 3.1. A fts (X, τ) is called

(1) r-fuzzy $\Re s$ compact if for every r-fuzzy regular semiopen cover $\{\lambda_i : i \in J\}$ of X, there exists a finite subset J_0 of J such that $\bigvee_{i\in J_0} \lambda_i = \overline{1}$.
(2) r-fuzzy weakly $\Re s$ compact if for every r-fuzzy regular semiopen cover $\{\lambda_i : i \in J\}$ of X, there exists a finite subset J_0 of J such that $\bigvee_{i\in J_0} I_\tau(\lambda_i, r) = \overline{1}$.
(3) r-fuzzy almost $\Re s$ compact if for every r-fuzzy regular semiopen cover $\{\lambda_i : i \in J\}$ of X, there exists a finite subset J_0 of J such that $\bigvee_{i\in J_0} C_\tau(\lambda_i, r) = \overline{1}$.

Remark. *(1) Every r-fuzzy weakly $\Re s$ compact is r-fuzzy $\Re s$ compact.*
(2) Every r-fuzzy $\Re s$ compact is r-fuzzy almost $\Re s$ compact.

From Theorem 2.3, we have the following theorem:

Theorem 3.1. *A fts (X,τ) is r-fuzzy $\Re s$-compact if and only if for each family $\{\lambda_i | i \in J\}$ of r-frso sets of X such that $\bigwedge_{i\in J} \lambda_i = \overline{0}$, there exists a finite subset J_0 of J such that $\bigwedge_{i\in J_0} \lambda_i = \overline{0}$.*

Theorem 3.2. *A fts (X,τ) is r-fuzzy weakly $\Re s$-compact if and only if for each family $\{\lambda_i | i \in J\}$ of r-frso sets of X such that $\bigwedge_{i\in J} \lambda_i = \overline{0}$, there exists a finite subset J_0 of J such that $\bigwedge_{i\in J_0} C_\tau(\lambda_i, r) = \overline{0}$.*

Proof. Suppose that (X,τ) is r-fuzzy weakly $\Re s$-compact. Let $\{\lambda_i | i \in J\}$ be a family of r-frso sets of X such that $\bigwedge_{i\in J} \lambda_i = \overline{0}$. Then by Theorem 2.3, $\{\overline{1} - \lambda_i | i \in J\}$ is a family of r-frso sets of X such that $\bigvee_{i\in J} \overline{1} - \lambda_i = \overline{1} - \bigwedge_{i\in J} \lambda_i = \overline{1}$. Since (X,τ) is r-fuzzy weakly $\Re s$ compact, there exists a finite subset J_0 of J such that $\bigvee_{i\in J_0} I_\tau(\overline{1} - \lambda_i, r) = \overline{1}$. Hence $\bigwedge_{i\in J_0} C_\tau(\lambda_i, r) = \overline{1} - (\bigvee_{i\in J_0} I_\tau(\overline{1} - \lambda_i, r) = \overline{0}$. □

Converse follows by reversing the previous arguments.

Theorem 3.3. *Let (X,τ) be a fts. Then the following are equivalent:*

(1) (X,τ) is r-fuzzy weakly $\Re s$-compact.
(2) For each family $\{\lambda_i | i \in J\}$ of r-frso sets of X such that $\bigwedge_{i\in J} \lambda_i = \overline{0}$, there exists a finite subset J_0 of J such that $\bigwedge_{i\in J_0} C_\tau(\lambda_i, r) = \overline{0}$.
(3) For each r-fuzzy regular closed cover $\{\lambda_i | i \in J\}$ of X, there exists a finite subset J_0 of J such that $\bigvee_{i\in J_0} I_\tau(\lambda_i, r) = \overline{1}$.

Proof. (1)⇒(2): Trivial.

(2)⇒(1): Let $\{\lambda_i | i \in J\}$ be a family of r-frso sets of X such that $\bigwedge_{i\in J} \lambda_i = \overline{0}$. Since λ_i is an r-frso set for each $i \in J$, $C_\tau(\lambda_i, r) = C_\tau(I_\tau(\lambda_i, r), r)$ for each $i \in J$. Since $\{I_\tau(\lambda_i, r) | i \in J\}$ is a family of r-fro sets of X such that $\bigwedge_{i\in J} I_\tau(\lambda_i, r) = \overline{0}$, by (2) there exists a finite subset J_0 of J such that $\bigwedge_{i\in J_0} C_\tau(\lambda_i, r) = \bigwedge_{i\in J_0} C_\tau(I_\tau(\lambda_i, r), r) = \overline{0}$. Thus (X,τ) is r-fuzzy weakly $\Re s$-compact.

(2)⇔(3): It is obvious. □

Theorem 3.4. *Let (X,τ) and (Y,σ) be two fts's and let $f : (X,\tau) \to (Y,\sigma)$ be surjective, fuzzy weakly open and fuzzy weakly continuous function. If (X,τ) is r-fuzzy weakly $\Re s$-compact, then so is (Y,σ).*

Proof. Let $\{\eta_i | i \in J\}$ be an r-fuzzy regular closed cover over Y. By Theorem 2.4, $\{f^1(\eta_i) | i \in J\}$ is an r-fuzzy regular closed cover of X. Since X is r- fuzzy weakly $\Re s$-compact, by Theorem 3.2, there exists a finite subset J_0 of J such that $\bigvee_{i\in J_0} I_\tau(f^{-1}(\eta_i), r) = \overline{1}$. From the surjectivity and fuzzy weakly openness of f, we have

$$\begin{aligned}
\overline{1} &= f(\textstyle\bigvee_{i\in J_0} (I_\tau(f^{-1}(\eta_i, r))))\\
&= \textstyle\bigvee_{i\in J_0} f(I_\tau(f^{-1}(\eta_i, r)))\\
&\leq \textstyle\bigvee_{i\in J_0} I_\sigma(f(C_\tau(I_\tau(f^{-1}(\eta_i), r), r)), r)\\
&= \textstyle\bigvee_{i\in J_0} (I_\sigma(f(f^{-1}(\eta_i)), r)\\
&= \textstyle\bigvee_{i\in J_0} I_\sigma(\eta_i, r).
\end{aligned}$$

Hence $\bigvee_{i\in J_0} I_\sigma(\eta_i, r) = \overline{1}$, and thus (Y,σ) is r-fuzzy weakly $\Re s$ compact. □

Theorem 3.5. *A fts (X,τ) is r-fuzzy almost $\Re s$ compact if and only if for each family $\{\lambda_i | i \in J\}$ of r-frso sets of X such that $\bigwedge_{i\in J} \lambda_i = \overline{0}$, there exists a finite subset J_0 of J such that $\bigwedge_{i\in J_0} I_\tau(\lambda_i, r) = \overline{0}$.*

Proof. Let (X,τ) be r-fuzzy almost $\Re s$-compact and let $\{\lambda_i | i \in J\}$ be a family of r-frso sets of X such that $\bigwedge_{i\in J} \lambda_i = \overline{0}$. Then $\{\overline{1} - \lambda_i | i \in J\}$ is a family of r-frso sets of X such that $\bigvee_{i\in J}\overline{1}-\lambda_i=\overline{1}-(\bigwedge_{i\in J}\lambda_i)=\overline{1}$. Since (X,τ) is r-fuzzy almost $\Re s$-compact, there exists a finite subset J_0 of J such that $\bigvee_{i\in J_0}C_\tau(\overline{1}-\lambda_i,r)=\overline{1}$. Hence $\bigwedge_{i\in J_0}I_\tau(\lambda_i,r)=\overline{1}-\bigvee_{i\in J_0}C_\tau(\overline{1}-\lambda_i,r)=\overline{0}$.

The converse can be proved similarly. □

Theorem 3.6. *Let (X,τ) be a fts. Then the following statements are equivalent:*

(1) (X,τ) is r-fuzzy almost $\Re s$-compact.

(2) For each family $\{\lambda_i|i\in J\}$ of r-fro sets of X such that $\bigwedge_{i\in J}\lambda_i=\overline{0}$, there exists a finite subset J_0 of J such that $\bigwedge_{i\in J_0}\lambda_i=\overline{0}$.

(3) For each r-fuzzy regular closed cover $\{\lambda_i|i\in J\}$ of X, there exists a finite subset J_0 of J such that $\bigvee_{i\in J_0}\lambda_i=\overline{1}$.

Proof. Straightforward. □

Definition 3.2. A fts (X,τ) is called an r-fuzzy S-closed if and only if for every an r-fuzzy semiopen cover $\{\lambda_i|i\in J\}$ of X, there exists a finite subset J_0 of J such that $\bigvee_{i\in J_0}C_\tau(\lambda_i,r)=\overline{1}$.

Theorem 3.7. *A fts (X,τ) is r-fuzzy almost $\Re s$-compact if and only if (X,τ) is r-fuzzy S-closed.*

Proof. Let (X,τ) be r-fuzzy S-closed. Since every r-frso set is r-fuzzy semiopen, (X,τ) is r-fuzzy almost $\Re s$-compact.

Conversely, suppose that (X,τ) is r-fuzzy almost $\Re s$-compact and let $\{\lambda_i|i\in J\}$ be an r-fuzzy semiopen cover of X. Then there exists $\mu_i\in I^X$ with $\tau(\mu_i)\ge r$, such that $\mu_i\le\lambda_i\le C_\tau(\mu_i,r)$, for each $i\in J$. We can easily show that $C_\tau(\mu_i,r)$ is an r-frc for each $i\in J$. Since $\mu_i\le\lambda_i\le C_\tau(\lambda_i,r)$, for each $i\in J$, $C_\tau(\mu_i,r)\le C_\tau(\lambda_i,r)\le C_\tau(C_\tau(\mu_i,r),r)$ for each $i\in J$. Thus $C_\tau(\lambda_i,r)=C_\tau(\mu_i,r)$ for each $i\in J$. Thus $\{C_\tau(\lambda_i,r)|i\in J\}$ is an r-fuzzy regular closed cover of X. Since (X,τ) is r-fuzzy almost $\Re s$-compact, there exists a finite subset J_0 of J such that $\bigvee_{i\in J_0}C_\tau(\lambda_i,r)=\overline{1}$. Hence (X,τ) is r-fuzzy S-closed. □

Theorem 3.8. *A fts (X,τ) is an r-fuzzy weakly $\Re s$-compact if and only if for every an r-fuzzy semiopen cover $\{\lambda_i|i\in J\}$ of X, there exists a finite subset J_0 of J such that $\bigvee_{i\in J_0}I_\tau(C_\tau(\lambda_i,r),r)=\overline{1}$.*

Proof. Similar to Theorem 3.7. □

Theorem 3.9. *Let (X,τ) and (Y,σ) be two fts's and let $f:(X,\tau)\to(Y,\sigma)$ be a surjective, fuzzy weakly open and fuzzy weakly continuous function. If (X,τ) is r-fuzzy almost $\Re s$-compact, then so is (Y,σ).*

Proof. Let $\{\eta_i|i\in J\}$ be an r-fuzzy regular closed cover of Y. By Theorem 2.4, $\{f^{-1}(\eta_i)|i\in J\}$ is an r-fuzzy regular closed cover of X. Since (X,τ) is r-fuzzy almost $\Re s$-compact, by Theorem 2.4, there exists a finite subset J_0 of J such that $\bigvee_{i\in J_0}f^{-1}(\eta_i)=\overline{1}$. From the surjectivity of f we have

$\overline{1}=f(\bigvee_{i\in J_0}f^{-1}(\eta_i))=\bigvee_{i\in J_0}f(f^{-1}(\eta_i))=\bigvee_{i\in J_0}\eta_i$.

Hence $\bigvee_{i\in J_0}\eta_i=\overline{1}$. Thus (Y,σ) is r-fuzzy almost $\Re s$-compact. □

Definition 3.3. A fts (X,τ) is called r-fuzzy extremally disconnected if $\tau(C_\tau(\lambda,r))\ge r$ for every $\lambda\in I^X$ with $\tau(\lambda)\ge r$.

Theorem 3.10. *Let (X,τ) and (Y,σ) be two fts, and let $f:(X,\tau)\to(Y,\sigma)$ be a surjective, fuzzy weakly open and fuzzy weakly continuous function. If (X,τ) is r-fuzzy extremally disconnected, then so is (Y,σ).*

Proof. Let $\lambda\in I^Y$ with $\sigma(\lambda)\geq r$. Then $\lambda=I_\sigma(\lambda,r)$. Hence $C_\tau(\lambda,r)$ is r-frc set. By Theorem 2.4, $f^{-1}(C_\sigma(\lambda,r))$ is r-frc, i.e., $f^{-1}(C_\sigma(\lambda,r))=C_\tau(I_\tau(f^{-1}(C_\sigma(\lambda,r)),r),r)$. Since (X,τ) is r-fuzzy extremally disconnected and $\tau(I_\tau(f^{-1}(C_\sigma(\lambda,r)),r))\geq r$ and $\tau(C_\tau(I_\tau(f^{-1}(C_\sigma(\lambda,r)),r),r))\geq r$. From the surjectivity and fuzzy weakly openness of f we have

$C_\sigma(\lambda,r)=f(f^{-1}(C_\sigma(\lambda,r)))$
$=f(C_\tau(I_\tau(f^{-1}(C_\sigma(\lambda,r)),r),r))$
$\leq I_\sigma(f(C_\tau(I_\tau(f^{-1}(C_\sigma(\lambda,r)),r),r)),r)$
$=I_\sigma(f(C_\tau(f^{-1}(C_\sigma(\lambda,r)),r)),r)$
$=I_\sigma(f(f^{-1}(C_\sigma(\lambda,r))),r)$
$=I_\sigma(C_\sigma(\lambda,r),r)$.

Hence $C_\sigma(\lambda,r)=I_\sigma(C_\sigma(\lambda,r),r)$ and so $\sigma(C_\sigma(\lambda,r))\geq r$. Thus (Y,σ) is r-fuzzy extremally disconnected. □

Theorem 3.11. *Let a fts (X,τ) be r-fuzzy extremally disconnected. If $\lambda\in I^X$ is r-frso, then $I_\tau(\lambda,r)=\lambda=C_\tau(\lambda,r)$.*

Proof. Let λ be an r-frso set. Then there exists an r-fro μ such that $\mu\leq\lambda\leq C_\tau(\mu,r)$. Since X is r-fuzzy extremally disconnected, $\mu=C_\tau(\mu,r)$. And we get $\mu=I_\tau(\mu,r)$, since μ is an r-fro set. Thus we have the following, $\mu=I_\tau(\mu,r)\leq I_\tau(\lambda,r)\leq\lambda\leq C_\tau(\lambda,r)\leq C_\tau(\mu,r)=\mu$. Hence $I_\tau(\lambda,r)=\lambda=C_\tau(\lambda,r)$. □

From the above theorem, we get the following:

Theorem 3.12. *Let a fts (X,τ) be r-fuzzy extremally disconnected. Then the following are equivalent:*

(1) (X,τ) is r-fuzzy weakly $\Re s$-compact.
(2) (X,τ) is r-fuzzy $\Re s$-compact.
(3) (X,τ) is r-fuzzy almost $\Re s$-compact.

Theorem 3.13. *For an r-fuzzy extremally disconnected fts (X,τ), the following are true:*

(1) r-fuzzy compactness implies r-fuzzy weakly $\Re s$-compactness.
(2) r-fuzzy nearly compactness implies r-fuzzy $\Re s$-compactness.
(3) r-fuzzy almost compactness implies r-fuzzy almost $\Re s$-compactness.

Proof. (2) Let (X,τ) be an r-fuzzy extremally disconnected and r-fuzzy nearly compact space, let $\{\lambda_i|i\in J\}$ be an r-fuzzy regular semiopen cover of X. Then there exists an r-fro set μ_i such that $\mu_i\leq\lambda_i\leq C_\tau(\mu_i,r)$ for each $i\in J$. Since (X,τ) is r-fuzzy extremally disconnected and $\mu_i=I_\tau(C_\tau(\mu_i,r))$ for each $i\in J$, $\lambda_i=I_\tau(\lambda_i,r)$ for each $i\in J$. Thus we get $\lambda_i=I_\tau(C_\tau(\lambda_i,r),r)$ for each $i\in J$ from Theorem 2.3. Hence (X,τ) is r-fuzzy $\Re s$-compact since X is r-fuzzy nearly compact.

(1) and (3) are similar to (2). □

Corollary 3.14. *If a fts (X,τ) is r-fuzzy extremally disconnected, then the following are equivalent:*

(1) r-fuzzy nearly compactness.
(2) r-fuzzy almost compactness.
(3) r-fuzzy S-closeness.

Proof. We get the results from Theorems 3.7, 3.12 and 3.13. □

4. r-FUZZY $\Re s$-CONNECTEDNESS

Definition 4.1. Let (X, τ) be a fts and $\lambda,\ \mu \in I^X$, $r \in I_0$. A r-fuzzy $\Re s$-separation on $\overline{1}$ is a pair of non null proper r-frso sets λ and μ such that $\lambda \wedge \mu = \overline{0}$ and $\lambda \vee \mu = \overline{1}$.

Definition 4.2. A fts (X, τ) is said to be r-fuzzy $\Re s$-connected if and only if there is no r-fuzzy $\Re s$-separation of $\overline{1}$. Otherwise, (X, τ) is said to be r-fuzzy $\Re s$-disconnected space.

Example 4.3. Let $X = \{a, b, c\}$, $\lambda, \mu, \delta \in I^X$, $r \in I_0$ are defined as $\lambda(a) = 0.2, \lambda(b) = 0.3, \lambda(c) = 0.4;\ \mu(a) = 0.6, \mu(b) = 0.3, \mu(c) = 0.4;\ \delta(a) = 0.7, \delta(b) = 0.4, \delta(c) = 0.5$. We define fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{3} & \text{if } \lambda = \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

For $r = \frac{1}{3}$, μ and δ are r-frso sets in (X, τ), $\mu \neq \overline{0}$, $\delta \neq \overline{0}$, $\mu \vee \delta \neq \overline{1}$ and $\mu \wedge \delta \neq \overline{0}$. Hence (X, τ) is r-fuzzy $\Re s$-connected.

Proposition 4.1. *A fts (X, τ) is a r-fuzzy $\Re s$-connected if and only if there exists no non-null r-frso sets $\lambda,\ \mu \in I^X$, $r \in I_0$ such that $\lambda = \overline{1} - \mu$.*

Proof. Necessity: Let λ and μ be two r-frso sets in (X, τ) such that $\lambda \neq \overline{0},\ \overline{1} - \mu \neq \overline{0}$ and $\lambda = \overline{1} - \mu$. Therefore $\overline{1} - \mu$ is a r-frsc set. Since $\lambda \neq \overline{0},\ \mu \neq \overline{1}$. This implies that μ is a proper fuzzy set which is both r-frso and r-frsc in (X, τ). Hence (X, τ) is not a r-fuzzy $\Re s$-connected space. But this is a contradiction to our hypothesis. Thus there exists no non-null r-frso sets λ and μ in (X, τ) such that $\lambda = \overline{1} - \mu$.

Sufficiency: Let λ be both r-frso and r-frsc in (X, τ) such that $\lambda \neq \overline{0},\ \lambda \neq \overline{1}$. Let $\overline{1} - \lambda = \mu$. Then μ is a r-frso set and $\overline{1} - \mu \neq \overline{1}$. This implies that $\mu = \overline{1} - \lambda \neq \overline{0}$, which is a contradiction to our hypothesis. Hence (X, τ) is a r-fuzzy $\Re s$-connected space. □

Proposition 4.2. *A fts (X, τ) is a r-fuzzy $\Re s$-connected space if and only if there exists no non-null r-frso sets $\lambda,\ \mu \in I^X$ with $r \in I_0$ such that $\lambda = \overline{1} - \mu$, $\mu = \overline{1} - RSC_\tau(\lambda)$ and $\lambda = \overline{1} - RSC_\tau(\mu)$.*

Proof. Necessity: Assume that there exists a fuzzy sets λ and μ such that $\lambda \neq \overline{0},\ \overline{1} - \mu \neq \overline{0}$, $\lambda = \overline{1} - \mu$, $\mu = \overline{1} - RSC_\tau(\lambda)$ and $\lambda = \overline{1} - RSC_\tau(\mu)$. Since $\overline{1} - RSC_\tau(\lambda)$ and $\overline{1} - RSC_\tau(\mu)$ are r-frso sets in (X, τ), λ and μ are r-frso sets in (X, τ). This implies (X, τ) is not a r-fuzzy $\Re s$-connected space, which is a contradiction. Thus there exists no non-null r-frso sets λ and μ in (X, τ) such that $\lambda = \overline{1} - \mu$, $\mu = \overline{1} - RSC_\tau(\lambda)$ and $\lambda = \overline{1} - RSC_\tau(\mu)$.

Sufficiency: Let λ be both r-frso and r-frsc in (X, τ) such that $\lambda \neq \overline{0}$, $\lambda \neq \overline{1}$. Now by taking $\overline{1} - \lambda = \mu$, we obtain a contradiction to our hypothesis. Hence (X, τ) is a r-fuzzy $\Re s$-connected space. □

Definition 4.4. A fts (X, τ) is said to be r-fuzzy C_5-disconnected if there exists fuzzy set $\lambda \in I^X$, $r \in I_0$, which is both r-fo and r-fc set such that $\lambda \neq \overline{0}$ and $\lambda \neq \overline{1}$. If (X, τ) is not r-fuzzy C_5-disconnected then it is said to be r-fuzzy C_5-connected.

Proposition 4.3. *Let (X, τ) and (Y, σ) be two fts's. Let $f : (X, \tau) \to (Y, \sigma)$ is a fuzzy regular semi continuous and surjective function. If (X, τ) is r-fuzzy $\Re s$-connected, then (Y, σ) is a r-fuzzy C_5-connected.*

Proof. Let (X, τ) is r-fuzzy $\Re s$-connected. Suppose (Y, σ) is not a r-fuzzy C_5-connected space, then there exists a proper fuzzy set $\lambda \in I^Y$, $r \in I_0$ which is both r-fo and r-fc set. Since f is a fuzzy regular semi continuous function, $f^{-1}(\lambda)$ is both r-frso and r-frsc in

(X, τ). But this is a contradiction to hypothesis. Hence (Y, σ) is a r-fuzzy C_5-connected space. $\square$

Definition 4.5. A fuzzy set in a fts (X, τ) is said to be r-frsco set, which is both r-frso and r-frsc set.

Definition 4.6. A fts (X, τ) is said to be r-fuzzy $\Re s$-C_5-disconnected if there exists r-frsco set $\lambda \in I^X$, $r \in I_0$ such that $\lambda \neq \overline{0}$ and $\lambda \neq \overline{1}$. If (X, τ) is not r-fuzzy $\Re s$-C_5-disconnected then it is said to be r-fuzzy $\Re s$-C_5-connected.

Proposition 4.4. *A fts (X, τ) is r-fuzzy $\Re s$-C_5 connected, then it is r-fuzzy $\Re s$-connected.*

Proof. Suppose that there exists non-null r-frso sets λ and μ such that $\lambda \vee \mu = \overline{1}$ and $\lambda \wedge \mu = \overline{0}$ (r-fuzzy $\Re s$-disconnected), then $\lambda = \lambda \vee \mu$ and $\lambda = \lambda \wedge \mu$. In other words, $\lambda = \overline{1} - \mu$. Hence λ is a r-frsco set which implies that (X, τ) is r-fuzzy $\Re s$-C_5-disconnected. $\square$

Remark. *The converse of the above Proposition need not be true as shown by the following example.*

Example 4.7. Let $X = \{a, b, c\}$, $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \mu, \delta \in I^X$ are defined as $\lambda_1(a) = 0.4, \lambda_1(b) = 0.5, \lambda_1(c) = 0.6$; $\lambda_2(a) = 0.4, \lambda_2(b) = 0.5, \lambda_2(c) = 0.4$; $\lambda_3(a) = 0.5, \lambda_3(b) = 0.5, \lambda_3(c) = 0.5$; $\lambda_4(a) = 0.5, \lambda_4(b) = 0.5, \lambda_4(c) = 0.6$; $\lambda_5(a) = 0.4, \lambda_5(b) = 0.5, \lambda_5(c) = 0.5$; $\mu(a) = 0.5, \mu(b) = 0.5, \mu(c) = 0.4$; $\delta(a) = 0.6, \delta(b) = 0.5, \delta(c) = 0.6$. We define fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{3} & \text{if } \lambda = \lambda_1, \\ \frac{1}{3} & \text{if } \lambda = \lambda_2, \\ \frac{1}{3} & \text{if } \lambda = \lambda_3, \\ \frac{1}{3} & \text{if } \lambda = \lambda_4, \\ \frac{1}{3} & \text{if } \lambda = \lambda_5, \\ 0 & \text{otherwise.} \end{cases}$$

For $r = \frac{1}{3}$, The fuzzy sets μ and δ are r-frso sets over $\overline{1}$ (since there exist r-fro set λ_1 such that $\lambda_1 \leq \mu \leq C_\tau(\lambda_1) = \overline{1} - \lambda_4$ and there exist r-fro set λ_4 such that $\lambda_4 \leq \delta \leq C_\tau(\lambda_4) = \overline{1} - \lambda_2$). Also, $\mu \wedge \delta = \mu \neq \overline{0}$, $\mu \vee \delta = \delta \neq \overline{1}$, hence (X, τ) is r-fuzzy $\Re s$-connected. But (X, τ) is r-fuzzy $\Re s$-C_5-disconnected, since λ_3 is both r-frso set and r-frsc set.

Proposition 4.5. *Let (X, τ) and (Y, σ) be fts's. Let $f : (X, \tau) \to (Y, \sigma)$ be a fuzzy regular semi irresolute and surjective function. If (X, τ) is r-fuzzy $\Re s$-connected, then (Y, σ) is r-fuzzy $\Re s$-connected.*

Proof. Assume that (Y, σ) is not r-fuzzy $\Re s$-connected. Thus there exists non-null r-frso sets $\lambda, \mu \in I^Y$, $r \in I_0$ such that $\lambda \vee \mu = \overline{1}$ and $\lambda \wedge \mu = \overline{0}$. Since f is fuzzy regular semi irresolute function, $\nu = f^{-1}(\lambda)$, $\eta = f^{-1}(\mu)$ are r-frso sets in (X, τ). From $\lambda \neq \overline{0}$, we get $\nu = f^{-1}(\lambda) \neq \overline{0}$. (If $f^{-1}(\lambda) \neq \overline{0}$, then $\lambda = f(f^{-1}(\lambda)) = f(\overline{0}) = \overline{0}$, which is a contradiction.) Similarly we obtain $\eta = \overline{0}$. Now, $\lambda \vee \mu = \overline{1}$ $f^{-1}(\lambda) \vee f^{-1}(\mu) = f^{-1}(\overline{1})$, $\nu \vee \eta = \overline{1}$, $\lambda \wedge \mu = \overline{0}$ $f^{-1}(\lambda) \wedge f^{-1}(\mu) = f^{-1}(\overline{0})$ $\nu \wedge \eta = \overline{0}$. This implies that $\nu \vee \eta = \overline{1}$ and $\nu \wedge \eta = \overline{0}$. Thus (X, τ) is r-fuzzy $\Re s$-connected, which is a contradiction to our hypothesis. Hence (Y, σ) is r-fuzzy $\Re s$-connected. $\square$

Proposition 4.6. *A fts (X, τ) is r-fuzzy $\Re s$-C_5-connected if and only if there exists no non-null r-frso sets $\lambda, \mu \in I^X$, $r \in I_0$ such that $\lambda = \overline{1} - \mu$.*

Proof. Suppose that λ and μ are r-frso sets in X such that $\lambda \neq \overline{0},\ \mu \neq \overline{0},\ \lambda = \overline{1} - \mu$. Since $\lambda = \overline{1} - \mu$, $\overline{1} - \mu$ is a r-frso set and μ is a r-frsc set. And $\lambda \neq \overline{0}$ implies $\mu \neq \overline{1}$. But this is a contradiction to the fact that (X, τ) is r-fuzzy $\Re s$-C_5-connected.

Conversely, let λ be both r-frso and r-frsc in X such that $\lambda \neq \overline{0},\ \lambda \neq \overline{1}$. Now take $\mu = \overline{1} - \lambda$. In this case μ is a r-frso set and $\lambda \neq \overline{1}$. Which implies that $\mu = \overline{1} - \lambda = \overline{0}$, which is a contradiction. □

Proposition 4.7. *A fts (X, τ) is r-fuzzy $\Re s$-C_5-connected if and only if there exists no non-null fuzzy sets $\lambda,\ \mu \in I^X,\ r \in I_0$ such that $\overline{1} - \lambda = \mu$, $\mu = \overline{1} - RSC_\tau(\lambda)$, $\lambda = \overline{1} - RSC_\tau(\mu)$.*

Proof. Assume that there exists a fuzzy sets λ and μ such that $\lambda \neq \overline{0},\ \mu \neq \overline{0}, \overline{1} - \lambda = \mu$, $\mu = \overline{1} - RSC_\tau(\lambda)$ and $\lambda = \overline{1} - RSC_\tau(\mu)$. Since $\overline{1} - RSC_\tau(\lambda)$ and $\overline{1} - RSC_\tau(\mu)$ are r-frso sets over X, λ and μ are r-frso sets in X, which is a contradiction.

Conversely, let λ be both r-frso and r-frsc in X such that $\lambda \neq \overline{0},\ \lambda \neq \overline{1}$. Taking $\mu = \overline{1} - \lambda$, we obtain a contradiction. □

Definition 4.8. A fts (X, τ) is said to be r-fuzzy $\Re s$-strongly connected if there exists no non-null r-frsc sets $\lambda,\ \mu \in I^X,\ r \in I_0$ such that $\lambda + \mu \leq \overline{1}$.

In otherwords, a fts (X, τ) is said to be r-fuzzy $\Re s$-strongly connected if there exists no non-null r-frsc sets $\lambda,\ \mu \in I^X,\ r \in I_0$ such that $\lambda \wedge \mu = \overline{1}$.

Proposition 4.8. *A fts (X, τ) is r-fuzzy $\Re s$-strongly connected if and only if there exists no non-null r-frso sets $\lambda,\ \mu \in I_X$ with $r \in I_0$ such that $\lambda \neq \overline{1},\ \mu \neq \overline{1}$ and $\lambda + \mu \geq \overline{1}$.*

Proof. Necessity: Let λ and μ are r-frso sets in (X, τ) such that $\lambda \neq \overline{1},\ \mu \neq \overline{1}$ and $\lambda + \mu \geq \overline{1}$. If we take $\nu = \overline{1} - \lambda$ and $\eta = \overline{1} - \mu$, then ν and η become r-frsc sets in X and $\nu \neq \overline{0},\ \eta \neq \overline{0}$ and $\nu + \eta \leq \overline{1}$. Which is a contradiction. Hence (X, τ) is r-fuzzy $\Re s$-strongly connected.

Sufficiency: Let λ and μ be non-null r-frsc sets in (X, τ) such that $\lambda + \mu \leq \overline{1}$. If $\nu = \overline{1} - \lambda$ and $\eta = \overline{1} - \mu$, then ν and η become r-frso sets in (X, τ) and $\nu \neq \overline{1}, \eta \neq \overline{1}$ and $\nu + \eta \geq \overline{1}$. Which is a contradiction. Thus there exists no non-null r-frso sets λ and μ in (X, τ) such that $\lambda \neq \overline{1}, \mu \neq \overline{1}$ and $\lambda + \mu \geq \overline{1}$. □

Proposition 4.9. *Let (X, τ) and (Y, σ) be fts's. Let $f : (X, \tau) \to (Y, \sigma)$ be a fuzzy regular semi irresolute and surjective function. If (X, τ) is r-fuzzy $\Re s$-strongly connected, then (Y, σ) is r-fuzzy $\Re s$-strongly connected.*

Proof. Suppose that (Y, σ) is not r-fuzzy $\Re s$-strongly connected. Then there exists non-null r-frsc sets ν_1 and ν_2 in (Y, σ) such that $\nu_1 \neq \overline{0}$, $\nu_2 \neq \overline{0}$, $\nu_1 + \nu_2 \leq \overline{0}$. Since f is fuzzy regular semi irresolute function, $f^{-1}(\nu_1)$, $f^{-1}(\nu_2)$ are r-frsc sets in (X, τ) and $f^{-1}(\nu_1) \wedge f^{-1}(\nu_2) = \overline{0}$, $f^{-1}(\nu_1) \neq \overline{0}$, $f^{-1}(\nu_2) \neq \overline{0}$. (If $f^{-1}(\nu_1) = \overline{0}$, then $f(f^{-1}(\nu_1)) = \nu_1$ which implies $f(\overline{0}) = \nu_1$. So $\overline{0} = \nu_1$ a contradiction.) Hence (X, τ) is r-fuzzy $\Re s$-strongly connected, a contradiction to our hypothesis. Thus (Y, σ) is r-fuzzy $\Re s$-strongly connected. □

Remark. *r-fuzzy $\Re s$-strongly connected does not imply r-fuzzy $\Re s$-C_5-connected.*

Example 4.9. In Example 4.7, (X, τ) is r-fuzzy $\Re s$-strongly connected, since there is no r-frsc sets $\lambda_1, \lambda_2, \lambda_1 + \lambda_2 \leq \overline{1}$. But (X, τ) is r-fuzzy $\Re s$-C_5-disconnected.

Remark. *r-fuzzy $\Re s$-C_5-connected does not imply r-fuzzy $\Re s$-strongly connected.*

Example 4.10. In Example 4.3, (X,τ) is r-fuzzy $\Re s$-C_5-strongly connected, since there is no fuzzy set λ is both r-frso and r-frsc set. But (X,τ) is not r-fuzzy $\Re s$-strongly connected, since there is the r-frsc sets λ and μ, $\lambda+\mu \leq \overline{1}$.

Definition 4.11. Let (X,τ) be fts, $\lambda,\ \mu \in I^X,\ r \in I_0$. The non-null fuzzy sets λ and μ are said to be

(1) r-fuzzy $\Re s$-weakly separated if $RSC_\tau(\lambda) \leq \overline{1}-\mu$ and $RSC_\tau(\mu) \leq \overline{1}-\lambda$.
(2) r-fuzzy $\Re s$-q-separated if $RSC_\tau(\lambda)\wedge\mu = \overline{0} = \lambda \wedge RSC_\tau(\mu)$.

Definition 4.12. A fts (X,τ) is said to be r-fuzzy $\Re s$-C_W-disconnected if there exists r-fuzzy $\Re s$-weakly separated non-null fuzzy sets λ and μ in X such that $\lambda \vee \mu = \overline{1}$.

Example 4.13. Let $X = \{a,b,c\}$, $\lambda,\ \mu \in I^X$, $r \in I_0$ are defined as $\lambda(a)=0, \lambda(b) = 1, \lambda(c) = 0;\ \mu(a) = 1, \mu(b) = 0, \mu(c) = 1$. We define fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0},\overline{1}\}, \\ \frac{1}{3} & \text{if } \lambda = \lambda, \\ \frac{1}{3} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

For $r = \frac{1}{3}$, the fuzzy sets λ and μ are r-frso sets in (X,τ), $RSC_\tau(\lambda) \leq \overline{1}-\mu$, $RSC_\tau(\mu) \leq \overline{1}-\lambda$. Hence λ and μ are r-fuzzy $\Re s$-weakly separated and $\lambda\vee\mu = \overline{1}$. Hence (X,τ) is r-fuzzy $\Re s$-C_W-disconnected.

Definition 4.14. A fts (X,τ) is said to be r-fuzzy $\Re s$-C_Q-disconnected if there exists r-fuzzy $\Re s$-q-separated non-null fuzzy sets λ and μ in X such that $\lambda\vee\mu = \overline{1}$.

Example 4.15. In Example 4.13, the fuzzy sets λ and μ are r-frso sets, $RSC_\tau(\lambda) = \overline{1}-\mu\wedge\mu = \overline{0}$ and $RSC_\tau(\mu) = \overline{1}-\lambda\wedge\lambda = \overline{0}$. Hence λ and μ are r-fuzzy $\Re s$-q-separated and $\lambda\vee\mu = \overline{1}$. Thus (X,τ) is r-fuzzy $\Re s$-C_Q-disconnected.

Remark. *A fts (X,τ) is said to be r-fuzzy $\Re s$-C_W-connected if and only if (X,τ) is r-fuzzy $\Re s$-C_Q-connected.*

Definition 4.16. Let (X,τ) be a fts and $Y \leq X$. Let λ^Y is defined as follows $\lambda^Y(x) = \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y \end{cases}$. Let $\tau_Y = \{\lambda^Y \wedge \mu : \tau(\mu) \geq r\}$, then the fuzzy topology τ_Y on Y is called fuzzy subspace topology and (Y,τ_Y) is called fuzzy subspace of (X,τ).

Definition 4.17. A fuzzy subspace (Y,τ_Y) of fts (X,τ) is said to be r-fuzzy $\Re s$-open (resp. r- fuzzy $\Re s$-closed, r-fuzzy $\Re s$-connected) subspace if $\lambda^Y \in FRSO(X)$ (resp. $\lambda^Y \in FRSC(X)$, λ^Y is r-fuzzy $\Re s$-connected).

Theorem 4.10. *Let (Y,τ_Y) be a r-fuzzy $\Re s$-connected subspace of fts (X,τ) such that $\gamma^Y\wedge\mu \in FRSO(X)$. If $\overline{1}$ has a r-fuzzy $\Re s$-separations λ and μ, then either $\gamma^Y \leq \lambda$ or $\gamma^Y \leq \mu$.*

Proof. Let $\lambda,\ \mu$ be r-fuzzy $\Re s$-separation on $\overline{1}$. By hypothesis, $\lambda\wedge\gamma^Y \in FRSO(X)$, $\mu\wedge\gamma^Y \in FRSO(X)$ and $[\lambda\wedge\gamma^Y]\vee[\mu\wedge\gamma^Y] = \gamma^Y$. Since γ^Y is r-fuzzy $\Re s$-connected. Then either $\lambda\wedge\gamma^Y = \overline{0}$ or $\mu\wedge\gamma^Y = \overline{0}$. Therefore, either $\gamma^Y \leq \lambda$ or $\gamma^Y \leq \mu$. □

Theorem 4.11. *If (X,τ_2) is a r-fuzzy $\Re s$-connected space and τ_1 is fuzzy coarser than τ_2, then (X,τ_1) is also a r-fuzzy $\Re s$-connected.*

Proof. Let $\lambda,\ \mu \in I^X$, $r \in I_0$ be r-fuzzy $\Re s$-separation on (X, τ_1). Then λ, μ are r-frso sets. Since $\tau_1 \leq \tau_2$. Then $\lambda,\ \mu$ in (X, τ_2) such that $\lambda,\ \mu$ is r-fuzzy $\Re s$-separation on (X, τ_2), which is a contradiction with the r-fuzzy $\Re s$-connectedness of (X, τ_2). Hence, (X, τ_1) is r-fuzzy $\Re s$-connected. □

Remark. *The converse of Theorem 4.11 is not true in general, as shown in the following example.*

Example 4.18. Let $X = \{a, b, c, d, e, f\}$, $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4 \in I^X$, $r \in I_0$ are defined as $\lambda_1(a) = 1, \lambda_1(b) = 1, \lambda_1(c) = 1, \lambda_1(d) = 0, \lambda_1(e) = 0, \lambda_1(f) = 0$; $\lambda_2(a) = 0.2, \lambda_2(b) = 0.3, \lambda_2(c) = 0.4, \lambda_2(d) = 0, \lambda_2(e) = 0, \lambda_2(f) = 0$; $\lambda_3(a) = 0, \lambda_3(b) = 0, \lambda_3(c) = 0, \lambda_3(d) = 1, \lambda_3(e) = 1, \lambda_3(f) = 1$; $\lambda_4(a) = 0.2, \lambda_4(b) = 0.3, \lambda_4(c) = 0.4, \lambda_4(d) = 1, \lambda_4(e) = 1, \lambda_4(f) = 1$. We define fuzzy topology $\tau_1, \tau_2 : I^X \to I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ 0 & \text{otherwise.} \end{cases} \quad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{3} & \text{if } \lambda = \lambda_1, \\ \frac{1}{3} & \text{if } \lambda = \lambda_2, \\ \frac{1}{3} & \text{if } \lambda = \lambda_3, \\ \frac{1}{3} & \text{if } \lambda = \lambda_4, \\ 0 & \text{otherwise.} \end{cases}$$

Let τ_1 be the indiscrete fuzzy $\Re s$-topology, then τ_1 is r-fuzzy $\Re s$-connected, on the other hand, Clearly, τ_2 defines a fuzzy topology on X such that $\tau_1 \leq \tau_2$. For $r = \frac{1}{3}$, λ_1 and λ_3 are r-frso sets in which form a r-fuzzy $\Re s$-separation of (X, τ_2) where $\lambda_1 \wedge \lambda_3 = \overline{0}$ and $\lambda_1 \vee \lambda_3 = \overline{1}$. Hence (X, τ_2) is r-fuzzy $\Re s$-disconnected.

Theorem 4.12. *A fuzzy subspace (Y, τ_Y) of a r-fuzzy $\Re s$-disconnected space (X, τ) is r-fuzzy $\Re s$-disconnected if $\gamma^Y \wedge \mu \in FRSO(X)$, $\forall \mu \in FRSO(X)$.*

Proof. Let (Y, τ_Y) be r-fuzzy $\Re s$-connected. Since (X, τ) is r-fuzzy $\Re s$-disconnected. Then there exists r-fuzzy $\Re s$-separation $\lambda,\ \mu$ on (X, τ). By hypothesis, $\lambda \wedge \gamma^Y \in FRSO(X)$, $\mu \wedge \gamma^Y \in FRSO(X)$ and $[\lambda \wedge \gamma^Y] \vee [\mu \wedge \gamma^Y] = \gamma^Y$, which is a contradiction with the r-fuzzy $\Re s$-connectedness of (Y, τ_Y). Therefore (Y, τ_Y) is r-fuzzy $\Re s$-disconnected. □

Remark. *A r-fuzzy $\Re s$-disconnectedness property is not hereditary property in general, as in the following example.*

Example 4.19. In Example 4.18, let $Y = \{a, b\} \leq X$. We consider the fuzzy set λ^Y of Y defined as follows, $\lambda^Y(a) = 1, \lambda^Y(b) = 1$. Then we define fuzzy subspace topology $\tau_Y : I^Y \to I$ as follows:

$$\tau_Y(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{3} & \text{if } \lambda = \lambda^Y \wedge \lambda_1, \\ \frac{1}{3} & \text{if } \lambda = \lambda^Y \wedge \lambda_2, \\ \frac{1}{3} & \text{if } \lambda = \lambda^Y \wedge \lambda_3, \\ \frac{1}{3} & \text{if } \lambda = \lambda^Y \wedge \lambda_4, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the collection $\tau_Y = \{\lambda^Y \wedge \mu : \tau(\mu) \geq r\}$ is a fuzzy subspace topology on Y in which there is no r-fuzzy $\Re s$-separation on (Y, τ_Y). Therefore, (Y, τ_Y) is r-fuzzy $\Re s$-connected at the time that (X, τ) is r-fuzzy $\Re s$-disconnected as shown in Example 4.18.

5. Conclusion

Sostak's fuzzy topology has been recently of major interest among fuzzy topologies. In this paper, we have introduced r-fuzzy regular semi compactness and gave basic definition and theorems of the concept. Also, we introduce r-fuzzy regular semi connectedness, r-fuzzy regular semi strongly connectedness and r-fuzzy regular semi-C_5-connectedness. Some interesting properties of these notions are studied.

6. Acknowledgements

The author would like to thank from the anonymous reviewers for carefully reading of the manuscript and giving useful comments, which will help us to improve the paper.

References

[1] B. Amudhambigai, M. K. Uma, and E. Roja. On rarely $\widetilde{g}$-continuous functions in smooth fuzzy topological spaces, The Journal of Fuzzy Mathematics, 20(2)(2012), 433–442.
[2] C. L. Chang. Fuzzy topological spaces, J. Math. Anal. Appl., 24(1968), 182–190.
[3] K. C. Chattopadhyay, and S. K. Samanta. Fuzzy topology, Fuzzy Sets and Systems, 54(1993), 207–212.
[4] K. C. Chattopadhyay, R. N. Hazra, and S. K. Samanta. Gradation of openness, Fuzzy Sets and Systems, 49(2) (1992), 237–242.
[5] C. K. Park, and W. K. Min. R-Semi-generalized fuzzy compactness, Korean J. Math., 16(3) 291–300.
[6] E. E. Kerre, and M. A. Fath Alla. On fuzzy regular semi open sets and αS^*-closed fuzzy topological spaces, The Journal of Fuzzy Mathematics, 11(1)(2003), 225–235.
[7] R. N. Hazra, S. K. Samanta, and K. C. Chattopadhyay. Fuzzy topology redefined, Fuzzy Sets and Systems, 45(1992), 79–82.
[8] A. S. Mashhour, M. H. Ghanim, and M. A. Fath Alla. α-separation axioms and α-compactness in fuzzy topological spaces, Rocky Moutain Journal of Mathematics, 16(3)(1986), 591–600.
[9] A. A. Ramadan, S. E. Abbas, and Y. C. Kim. Fuzzy irresolute mappings in smooth fuzzy topological spaces, The Journal of Fuzzy Mathematics, 9(4)(2001), 865–877.
[10] Seok Jong Lee, and Eun Pyo Lee. Fuzzy r-regular open sets and fuzzy almost r-continuous maps, Bull. Korean Math. Soc., 39(3)(2002), 441–453.
[11] A. P. $\check{S}$ostak. On a fuzzy topological structure, Rend. Circ. Matem. Palermo Ser II, 11(1986), 89–103.
[12] A. Vadivel, and E. Elavarasan. r-fuzzy regular semi open sets in smooth topological spaces, Sahand Communications i Mathematical Analysis, 6(1)(2017), 1–17.
[13] A. M. Zahran. Fuzzy regular semi open sets and αS-closed spaces, The Journal of Fuzzy Mathematics, 2(4)(1997), 579–586.

E. Elavarasan
Department of Mathematics, Shree Raghavendra Arts and Science College (Affiliated to Thiruvalluvar University), Keezhamoongiladi, Chidambaram-608102, Tamil Nadu, India.
Email address: maths.aras@gmail.com

ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 3, Number 4 (2020), 285-292
ISSN: 2582-0818
© http://www.technoskypub.com

STABILITY OF FINITE VARIABLE QUARTIC FUNCTIONAL EQUATION IN CLASSICAL METHODS

K. TAMILVANAN, K. LOGANATHAN* AND G. BALASUBRAMANIAN

ABSTRACT. In this work, we investigate the Hyers-Ulam stability by using direct and fixed point methods for the quartic functional equation

$$\begin{aligned}\sum_{b=1}^{p}\phi\left(-v_b+\sum_{a=1;a\neq b}^{p}v_a\right) = \quad & 4\sum_{1\leq a<b<c\leq p}\phi\left(v_a+v_b+v_c\right)+(-4p+14)\sum_{a=1;a\neq b}^{p}\phi\left(v_a+v_b\right)\\ & +2\sum_{a=1;a\neq b}^{p}\phi\left(v_a-v_b\right)+(p-8)\phi\left(\sum_{a=1}^{p}v_a\right)+\sum_{b=1}^{p}\phi\left(2v_b\right)\\ & +(2p^2-14p+14)\sum_{a=1}^{p}\phi(v_a)\end{aligned}$$

for positive integer $p \geq 3$.

1. INTRODUCTION

The stability problem of a functional equation became first posed by way of Ulam [13] regarding the stability of group homomorphism which become responded by means of Hyers [6] for Banach spaces. Hyers theorem became generalized through Aoki [2] for additive mapping and through Rassias [11] for linear mappings by using considering an unbounded Cauchy difference. Rassias [11] has provided plenty of have an impact on in the improvement of what we name generalized Hyers-Ulam stability of functional equations.

A generalization of the Th. M. Rassias theorem became acquired with the aid of P. Gavruta cite4through replacing the unbounded Cauchy difference by using a wellknown control function within the spirit of Rassias technique. Won-Gil Park and Jae-Hyeong Bae cite11, delivered the subsequent functional equation

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)+24f(y)-6f(x) \tag{1.1}$$

and that they mounted the general solution of the functional equation (1.1). It is straightforward to look that the function $f(x)=x^4$ is a solution of the functional equation (ref1.1). Therefore, it is natural that (1.1) is referred to as a quartic functional equation and each

2010 *Mathematics Subject Classification.* 39B52, 39B72, 39B82.
Key words and phrases. Quartic Functional Equation; Fixed Point; Hyers-Ulam Stability; Banach Space.
Received: October 02, 2020. Accepted: December 12, 2020. Published: December 31, 2020.
*Corresponding author.

solution of the quartic functional equation is stated to be quartic mapping. Numerous authors are inspect the stableness for the functional equations in banach and numerous spaces which offers an concept to develop this paper (see [1, 3, 4, 6, 7, 9, 10, 12]).

The aim of this paper is to obtain the Hyers-Ulam stability by using direct and fixed point methods for the quartic functional equation

$$\begin{aligned}\sum_{b=1}^{p}\phi\left(-v_b+\sum_{a=1;a\neq b}^{p}v_a\right) = \quad & 4\sum_{1\le a<b<c\le p}\phi\left(v_a+v_b+v_c\right)+(-4p+14)\sum_{a=1;a\neq b}^{p}\phi\left(v_a+v_b\right)\\ & +2\sum_{a=1;a\neq b}^{p}\phi\left(v_a-v_b\right)+(p-8)\phi\left(\sum_{a=1}^{p}v_a\right)+\sum_{b=1}^{p}\phi\left(2v_b\right)\\ & +(2p^2-14p+14)\sum_{a=1}^{p}\phi(v_a) \qquad (1.2)\end{aligned}$$

for positive integer $p\geq 3$ in Banach space.

Theorem (Alternative of fixed point): Suppose that for a complete generalized metric space (A,d) and a strictly contractive mapping $\Gamma : A \to A$ with Lipschitz constant L. Then, for each given element $u\in A$ either

(B1) $d(\Gamma^i u, \Gamma^{i+1}u) = +\infty \quad \forall i\geq 0$, or

(B2) There exists natural number i_0 such that

i) $d(\Gamma^i u, \Gamma^{i+1}u) < \infty \quad \forall i \geq i_0$;

ii) the sequence $(\Gamma^i u)$ is convergent to a fixed point v^* of Γ;

iii) v^* is the unique fixed point of Γ in the set $B=\{v\in A; d(\Gamma^{i_0}u, v)<\infty\}$;

iv) $d(v^*,v)\leq \frac{1}{1-L}d(v,\Gamma v) \quad \forall v\in B$.

Consider E be a normed space and F be a Banach space. For notational handiness, we define a function $\phi : E\to F$ by

$$\begin{aligned}D\phi\left(v_1,v_2,\cdots,v_p\right) \quad &= \sum_{b=1}^{p}\phi\left(-v_b+\sum_{a=1;a\neq b}^{p}v_a\right)-4\sum_{1\le a<b<c\le p}\phi\left(v_a+v_b+v_c\right)\\ &-(-4p+14)\sum_{a=1;a\neq b}^{p}\phi\left(v_a+v_b\right)-2\sum_{a=1;a\neq b}^{p}\phi\left(v_a-v_b\right)\\ &-(p-8)\phi\left(\sum_{a=1}^{p}v_a\right)-\sum_{b=1}^{p}\phi\left(2v_b\right)-(2p^2-14p+14)\sum_{a=1}^{p}\phi(v_a)\end{aligned}$$

for all $v_1,v_2,\cdots,v_p\in E$.

2. Stability Result for (1.2) in Banach space using Direct Method

Theorem 2.1. *Let $i\in\{-1,1\}$. Let $\zeta : E^p\to[0,\infty)$ be a function such that $\sum_{r=0}^{\infty}\frac{\zeta(2^{ri}v_1,2^{ri}v_2,\cdots,2^{ri}v_p)}{2^{4ri}}$ converges in $\mathbb{R}$ and*

$$\lim_{r\to\infty}\frac{\zeta(2^{ri}v_1,2^{ri}v_2,\cdots,2^{ri}v_p)}{2^{4ri}}=0 \quad \forall\, v_1,v_2,\cdots,v_p\in E. \qquad (2.1)$$

If $\phi : E\to F$ be a function fulfils

$$||D\phi(v_1,v_2,\cdots,v_p)||\leq\zeta(v_1,v_2,\cdots,v_p) \quad \forall\, v_1,v_2,\cdots,v_p\in E, \qquad (2.2)$$

then there exist a unique quartic function $Q_4 : E \to F$ which fulfils (1.2) and

$$||\phi(v) - Q_4(v)|| \leq \frac{1}{16} \sum_{r=\frac{1-i}{2}}^{\infty} \frac{\nu(2^{ri})}{2^{4ri}} \tag{2.3}$$

where $\nu(v) = \zeta(v, 0, \cdots, 0) \quad \forall v \in E$. The function Q_4 is given by

$$Q_4(v) = \lim_{r\to\infty} \frac{\phi(2^{ri}v)}{2^{4ri}} \quad \forall v \in E. \tag{2.4}$$

Proof. Assume that $i = 1$. Replacing $(v_1, v_2, \cdots, v_p)$ by $(v, 0, \cdots, 0)$ in (2.2), we get

$$\|16\phi(v) - \phi(2v)\| \leq \zeta(v, 0, \cdots, 0) \quad \forall v \in E. \tag{2.5}$$

It follows from (2.5) that

$$\|\frac{\phi(2v)}{2^4} - \phi(v)\| \leq \frac{1}{2^4}\zeta(v, 0, \cdots, 0) \quad \forall v \in E. \tag{2.6}$$

Switching v through $2v$ and dividing by 2^4 in (2.6), we arrive

$$\|\frac{\phi(2^2v)}{2^8} - \frac{\phi(2v)}{2^4}\| \leq \frac{1}{2^8}\zeta(2v, 0, \cdots, 0) \quad \forall v \in E. \tag{2.7}$$

Adding (2.6) and (2.7), we have

$$\|\frac{\phi(2^2v)}{2^4} - \phi(v)\| \leq \frac{1}{2^4}\left(\zeta(v, 0, \cdots, 0) + \frac{\zeta(2v, 0, \cdots, 0)}{2^4}\right) \quad \forall v \in E.$$

In general for any integer $s > 0$, one can easy to verify that

$$\|\frac{\phi(2^sv)}{2^{4s}} - \phi(v)\| \leq \frac{1}{2^4}\sum_{r=0}^{\infty} \frac{\nu(2^rv)}{2^{4r}} \quad \forall v \in E. \tag{2.8}$$

In order to show the convergence of the sequence $\{\frac{\phi(2^sv)}{2^{4s}}\}$, replacing v by 2^tv and dividing 2^{4t} in (2.8), for $s, t > 0$, we get

$$\|\frac{\phi(2^{s+t}v)}{2^{4(s+t)}} - \frac{\phi(2^tv)}{2^{4v}}\| \leq \frac{1}{2^4}\sum_{r=0}^{s-1} \frac{\nu(2^{r+t}v)}{2^{4(r+t)}} \to 0 \ \ as \ \ t \to \infty \tag{2.9}$$

for all $v \in E$. Therefore, $\{\frac{\phi(2^sv)}{2^{4s}}\}$ is a Cauchy sequence. As F is complete, there exists a mapping $Q_4 : E \to F$ such that

$$Q_4(v) = \lim_{s\to\infty} \frac{\phi(2^sv)}{2^{4s}} \quad \forall v \in E.$$

Passing $s \to \infty$ in (2.8) we see that (2.3) holds for $v \in E$. To show that Q_4 fulfils (1.2), switching $(v_1, v_2, \cdots, v_p)$ by $(2^tv, 2^tv, \cdots, 2^tv)$ and dividing 2^{4t} in (2.2), we arrive

$$\frac{1}{2^{4t}}\|Q_4(2^tv, 2^tv, \cdots, 2^tv)\| \leq \frac{1}{2^{4t}}\zeta(2^tv, 2^tv, \cdots, 2^tv) \quad \forall v_1, v_2, \cdots, v_p \in E.$$

Letting $t \to \infty$ in above inequality and using the definition of $Q_4(v)$, we see that $Q_4(v_1, v_2, \cdots, v_p) = 0$. Hence Q_4 satisfies (1.2) for all $v \in E$. To show that Q_4 is unique. Let R_4 be the another quartic mapping fulfilling (1.2) and (2.3), then

$$\begin{aligned}\|Q_4(v) - R_4(v)\| &\leq \frac{1}{2^{4t}}\{\|Q_4(2^tv) - \phi(2^t)\| + \|\phi(2^tv) - R_4(2^tv))\|\} \\ &\leq \frac{1}{2^4}\sum_{r=0}^{\infty} \frac{\nu(2^{r+t}v)}{2^{4(r+t)}} \to 0 \ \ as \ \ t \to \infty\end{aligned}$$

for all $v \in E$. Hence Q_4 is unique. Similarly, we can derive the stability results for $i = -1$. $\square$

Corollary 2.2. *Let a and b be a non-negative real numbers. Let $\phi : E \to F$ be a function fulfilling*

$$||D\phi(v_1, v_2, v_3, \cdots, v_p)|| \leq \begin{cases} a, \\ a(\sum_{j=1}^{p} ||v_j||^b), \\ a(\prod_{j=1}^{p} ||v_j||^b + \sum_{j=1}^{p} ||v_j||^{pb}), \end{cases}$$

for all $v_1, v_2, \cdots, v_p \in E$. Then there exists a unique quartic function $Q_4 : E \to F$ such that

$$||\phi(v) - Q_4(v)|| \leq \begin{cases} \frac{a}{|15|}, \\ \frac{a||v||^b}{|2^4 - 2^b|}; & b \neq 4, \\ \frac{a||v||^{pb}}{|2^4 - 2^{pb}|}; & b \neq \frac{4}{p}, \end{cases}$$

for all $v \in E$.

3. Stability Result for (1.2) in Banach space using Fixed Point Method

Theorem 3.1. *Let $\phi : E \to F$ be a mapping for which there exists a function $\zeta : E^p \to [0, \infty)$ with the condition*

$$lim_{r\to\infty} \frac{\zeta(\tau_\delta^r v_1, \tau_\delta^r v_2, \cdots, \tau_\delta^r v_p)}{\tau_\delta^{4r}} = 0 \tag{3.1}$$

where

$$\tau_\delta = \begin{cases} 2, \ if & \delta = 0; \\ \frac{1}{2}, \ if & \delta = 1; \end{cases}$$

such that the functional inequality

$$||D\phi(v_1, v_2, \cdots, v_p)|| \leq \zeta(v_1, v_2, \cdots, v_p) \quad \forall v_1, v_2, \cdots, v_p \in E. \tag{3.2}$$

If there exist $L = L(\delta)$ such that the function

$$v \to \Upsilon(v) = \zeta\left(\frac{v}{2}, 0, \cdots, 0\right)$$

has the property,

$$\frac{1}{\tau_\delta^4} \Upsilon(\tau_\delta v) = L\Upsilon(v) \quad \forall v \in E. \tag{3.3}$$

Then there exists a unique quartic function $Q_4 : E \to F$ fulfilling (1.2) and

$$||\phi(v) - Q_4(v)|| \leq \frac{L^{1-\delta}}{1-L} \Upsilon(v) \tag{3.4}$$

holds for all $v \in E$.

Proof. Suppose $d = \{s/s : E \to F, s(0) = 0)\}$ and define the generalized metric on Φ. $d(s,t) = \inf\{r \in (0,\infty) : ||s(v) - t(v)|| \leq r\Upsilon(v), v \in E\}$. It is easy to see that (Φ, d) is complete. Define $\Psi : \Phi \to \Phi$ by

$$\Psi s(v) = \frac{1}{\tau_\delta^4} s(\tau_\delta v) \quad \forall v \in \Phi.$$

Now $s, t \in \Phi$,

$$d(s,t) \leq r \Rightarrow ||s(v) - t(v)|| \leq r\Upsilon(v) \quad \forall v \in E.$$

$$\Rightarrow ||\frac{1}{\tau_\delta^4}s(\tau_\delta v) - \frac{1}{\tau_\delta^4}t(\tau_\delta v)|| \leq \frac{1}{\tau_\delta^4}r\Upsilon(\tau_\delta v) \quad \forall v \in E.$$
$$\Rightarrow ||\Psi s(v) - \Psi t(v)|| \leq r\Upsilon(v) \quad \forall v \in E.$$
$$\Rightarrow d(\Psi s, \Psi t) \leq rL.$$

This implies $d(\Psi s, \Psi t) \leq Ld(s,t) \quad \forall s,t \in \Phi$. (i,e.,) Ψ is strictly contractive mapping on with Lipschtiz constant L. Switching $(v_1, v_2, \cdots, v_p)$ by $(v, 0, \cdots, 0)$ in (3.2), we obtain

$$||\phi(2v) - 16\phi(v)|| \leq \zeta(v, 0, \cdots, 0) \quad \forall v \in E. \tag{3.5}$$

It is follows from (3.5) that

$$||\phi(v) - \frac{\phi(2v)}{16}|| \leq \frac{\zeta(v, 0, \cdots, 0)}{16} \quad \forall v \in E. \tag{3.6}$$

Utilizing (3.3) for $\delta = 0$, we have

$$||\phi(v) - \frac{\phi(2v)}{16}|| \leq \Upsilon(v) \quad \forall v \in E.$$
$$i.e., d(\phi, \Psi\phi) \leq 1 \Rightarrow d(\phi, \Psi\phi) \leq 1 = L = L^1 < \infty.$$

Again interchanging $v = \frac{v}{2}$ in (3.5) and (3.6), we get

$$||\phi(v) - 16\phi\left(\frac{v}{2}\right)|| \leq \zeta\left(\frac{v}{2}, 0, \cdots, 0\right)$$

and

$$||\phi(v) - 16\phi\left(\frac{v}{2}\right)|| \leq \zeta\left(\frac{v}{2}, 0, \cdots, 0\right) \quad \forall v \in E. \tag{3.7}$$

Utilizing (3.3) for $\delta = 0$, we have

$$||\phi(v) - \frac{\phi(2v)}{16}|| \leq L\Upsilon(v) \quad \forall v \in E. \tag{3.8}$$

(i.e.,) $d(\phi, \Psi\phi) \leq 1 \Rightarrow d(\phi, \Psi\phi) \leq 1 = L^0 < \infty$. In above case, we arrive

$$d(\phi, \Psi\phi) \leq L^{1-\delta}.$$

Therefore, $(B_2(i))$holds. By $(B_2(ii))$, it follows that there exists a fixed point Q_4 of Ψ in E, such that

$$Q_4(v) = lim_{r\to\infty}\frac{\phi(\tau_\delta^r v)}{\tau_\delta^{4r}} \quad \forall v \in E. \tag{3.9}$$

In order to prove $Q_4 : E \to F$ is quartic. Interchanging $(v_1, v_2, \cdots, v_p)$ through $(\tau_\delta^r v_1, \tau_\delta^r v_2, \cdots, \tau_\delta^r v_p)$ in (3.2) and dividing by τ_δ^{4r}, it follows from (3.1) and (3.9), we see that Q_4 fulfils (1.2) for all $v_1, v_2, \cdots, v_p \in E$. Hence Q_4 fulfils (1.2). By $(B_2(iii))$, Q_4 is the unique fixed point of Ψ in the set, $F = \{\phi \in \Phi; d(\Psi\phi, Q_4) < \infty\}$. Utilizing the fixed point alternative result, Q_4 is the unique function such that,

$$||\phi(v) - Q_4(v)|| \leq r\Upsilon(v) \quad \forall v \in E, r > 0.$$

Finally, by $(B_2(iv))$, we reach

$$d(\phi, Q_4) \leq \frac{1}{1-L}d(\phi, \Psi\phi)$$
$$(i.e.,)\ d(\phi, Q_4) \leq \frac{L^{1-\delta}}{1-L}.$$

Hence, we conclude that

$$||\phi(v) - Q_4(v)|| \leq \frac{L^{1-\delta}}{1-L}\Upsilon(v) \quad \forall v \in E.$$

□

Corollary 3.2. *Let $\phi : E \to F$ be a mapping and there exists a real numbers a and b such that*

$$||D\phi(v_1, v_2, \cdots, v_p)|| \leq \begin{cases} a, \\ a(\sum_{j=1}^{p} ||v_j||^b), \\ a(\prod_{j=1}^{p} ||v_j||^b + \sum_{j=1}^{p} ||v_j||^{pb}), \end{cases}$$

for all $v_1, v_2, \cdots, v_p \in E$. Then there exist a unique quartic function $Q_4 : E \to F$ such that

$$||\phi(v) - Q_4(v)|| \leq \begin{cases} \frac{a}{|15|}, \\ \frac{a||v||^b}{|2^4 - 2^b|}; \quad b \neq 4, \\ \frac{a||v||^{pb}}{|2^4 - 2^{pb}|}; \quad b \neq \frac{4}{p}, \end{cases}$$

for all $v \in E$.

Proof. Setting

$$\zeta(v_1, v_2, \cdots, v_p) \leq \begin{cases} a, \\ a(\sum_{j=1}^{p} ||v_j||^b), \\ a(\prod_{j=1}^{p} ||v_j||^b + \sum_{j=1}^{p} ||v_j||^{pb}), \end{cases}$$

for all $v_1, v_2, \cdots, v_p \in E$. Now

$$\frac{\zeta(\tau_\delta^r v_1, \tau_\delta^r v_2, \cdots, \tau_\delta^r v_p)}{\tau_\delta^{4r}} \leq \begin{cases} \frac{a}{\tau_\delta^{4r}}, \\ \frac{a}{\tau_\delta^{4r}} \left\{ \sum_{j=1}^{p} ||\tau_\delta^r v_j||^b \right\}, \\ \frac{a}{\tau_\delta^{4r}} \left\{ \prod_{j=1}^{p} ||\tau_\delta^r v_j||^{pb} + \sum_{j=1}^{p} ||\tau_\delta^r v_j||^{pb} \right\}, \end{cases}$$

$$= \begin{cases} \longrightarrow \quad 0 \quad as \quad r \longrightarrow \infty, \\ \longrightarrow \quad 0 \quad as \quad r \longrightarrow \infty, \\ \longrightarrow \quad 0 \quad as \quad r \longrightarrow \infty, \end{cases}$$

i.e., (3.5) is holds. But we have $\Upsilon(v) = \zeta\left(\frac{v}{2}, 0, \cdots, 0\right)$. Hence

$$\Upsilon(v) = \zeta\left(\frac{v}{2}, 0, \cdots, 0\right) = \begin{cases} a, \\ \frac{a||v||^b}{2^b}, \\ \frac{a||v||^{pb}}{5^{pb}}, \end{cases}$$

$$\frac{1}{\tau_\delta^4}\Upsilon(\tau_\delta v) = \begin{cases} \frac{a}{\tau_\delta^4}, \\ \frac{1}{\tau_\delta^4}\frac{a||v||^b}{2^b}, \\ \frac{1}{\tau_\delta^4}\frac{a||v||^{pb}}{2^{pb}}, \end{cases}$$

$$= \begin{cases} \tau_\delta^{-4}\Upsilon(v), \\ \tau_\delta^{b-4}\Upsilon(v), \\ \tau_\delta^{pb-4}\Upsilon(v), \end{cases}$$

for all $v \in E$. Hence the inequality (1.2) holds for
$L = 2^{-4}$ if $\delta = 0$ and $L = \frac{1}{2^{-4}}$ if $\delta = 1$.
$L = 2^{b-4}$ for $b < 4$ if $\delta = 0$ and $L = \frac{1}{2^{b-4}}$ for $b > 4$ if $\delta = 1$.
$L = 2^{pb-4}$ for $b < \frac{4}{p}$ if $\delta = 0$ and $L = \frac{1}{2^{pb-4}}$ for $b > \frac{4}{p}$ if $\delta = 1$.
Now, from (3.5) we prove the following cases:

Case1:$L = 2^{-4}$ if $\delta = 0$.

$$||\phi(v) - Q_4(v)| \leq \frac{L^{1-\delta}}{1-L}\Upsilon(v) = \frac{(2^{-4})}{1-2^{-4}}a = \frac{a}{15}.$$

Case2:$L = \frac{1}{2^{-4}}$ if $\delta = 1$.

$$||\phi(v) - Q_4(v)|| \leq \frac{L^{1-\delta}}{1-L}\Upsilon(v) = \frac{1}{1-2^{4}}a = \frac{-a}{16}.$$

Case3:$L = 2^4$ for $b < 4$ if $\delta = 0$.

$$||\phi(v) - Q_4(v)|| \leq \frac{L^{1-\delta}}{1-L}\Upsilon(v) = \frac{2^{b-4}}{1-2^{b-4}}\frac{a||v||^b}{2^b} = \frac{a||v||^b}{2^4-2^b}.$$

Case4:$L = \frac{1}{2^{b-4}}$ for $b > 4$ if $\delta = 1$.

$$||\phi(v) - Q_4(v)|| \leq \frac{L^{1-\delta}}{1-L}\Upsilon(v) = \frac{1}{1-\frac{1}{2^{b-4}}}\frac{a||v||^b}{2^b} = \frac{a||v||^b}{2^b-2^4}.$$

Case5:$L = 2^{pb-4}$ for $b < \frac{4}{p}$ if $\delta = 0$.

$$||\phi(v) - Q_4(v)|| \leq \frac{L^{1-\delta}}{1-L}\Upsilon(v) = \frac{2^{pb-4}}{1-2^{pb-4}}\frac{a||v||^{pb}}{2^{pb}} = \frac{a||v||^{pb}}{2^4-2^{pb}}.$$

Case6:$L = \frac{1}{2^{pb-4}}$ for $b > \frac{4}{p}$ if $\delta = 1$.

$$||\phi(v) - Q_4(v)|| \leq \frac{L^{1-\delta}}{1-L}\Upsilon(v) = \frac{1}{1-\frac{1}{2^{pb-4}}}\frac{a||v||^{pb}}{2^{pb}} = \frac{a||v||^{pb}}{2^{pb}-2^4}.$$

Hence the proof. □

REFERENCES

[1] J. Aczel, and J. Dhombres. Functional Equations in Several Variables, Cambridge univ. press, (1989).
[2] T. Aoki. On the stability of the linear transformation in Banach Spaces, J. Math. Soc. Japan 2 (1950), p. 64–66.
[3] J. H. Bae, and W. G. Park. On the solution of a bi-Jensen functional equation and its stability, Bull. Korean Math. Soc. 43(3) (2006), p. 499–507.
[4] A. Bodaghi. Stability of a quartic functional equation, Sci. World J. Art. ID 752146, (2014) 9 pages.
[5] P. Gavruta. A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. 184 (1994), p. 431–436.
[6] D. H. Hyers. On the stability of the linear functional equation, Proc. Natl. Acad. Sci. U.S.A. 27(1941), p. 222–224.
[7] Y. H. Lee, and K. W. Jun. On the stability of approximately additive mappings , Proc. Amer. Math. Soc. 128(2000), p. 1361–1369.
[8] W. G. Park, and J. H. Bae. On the stability a bi-quartic functional equation , Nonlinear Anal. 62(4)(2005), p. 643-654.
[9] C. Park, K. Tamilvanan, G. Balasubramanian, B. Noori, and A. Najati. On a functional equation that has the quadratic-multiplicative property, Open Mathematics, 18(2020), p. 837–845.
[10] C. Park, K. Tamilvanan, B. Noori, M. B. Moghimi, and A. Najati. Fuzzy normed spaces and stability of a generalized quadratic functional equation, AIMS Mathematics, 5(6)(2020), p. 7161-7174.
[11] Th. M. Rassias. On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72(1978), p. 297–300.
[12] K. Tamilvanan, J. R. Lee, and C. Park. Hyers-Ulam stability of a finite variable mixed type quadratic-additive functional equation in quasi-Banach spaces, AIMS Math., 5(6)(2020), p. 5993-6005.
[13] S. M. Ulam. Problems in Modern Mathematics, Science Editions, Wiley, New York (1964).

K. TAMILVANAN
GOVERNMENT ARTS COLLEGE FOR MEN, KRISHNAGIRI-635 001, TAMIL NADU, INDIA.
Email address: tamiltamilk7@gmail.com

K. LOGANATHAN
DEPARTMENT OF MATHEMATICS, ERODE ARTS & SCIENCE COLLEGE, ERODE - 638 009, TAMIL NADU, INDIA.
Email address: loganathankaruppusamy304@gmail.com

G. BALASUBRAMANIAN
GOVERNMENT ARTS COLLEGE FOR MEN, KRISHNAGIRI-635 001, TAMIL NADU, INDIA.
Email address: gbs_geetha@yahoo.com

ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 3, Number 4 (2020), 293-302
ISSN: 2582-0818

STRUCTURE AND APPLICATION OF HCA IN IMAGE ANALYSIS

M. RAJASEKAR AND R. ANBU*

ABSTRACT. In this paper, we study the neighborhood structure of hexagonal cellular automata with null boundary conditions over the field $\mathbb{Z}_2$. Rule matrix with null boundary condition and application of $\mathcal{HCA}$ in the field of image analysis are studied.

1. INTRODUCTION

The concept of Cellular Automata($\mathcal{CA}$) was initiated in the early 1950's by John Von Neumann and Stan Ulam [8, 10]. Afterwards, Stephen Wolfram developed the $\mathcal{CA}$ theory [11].

The Hexagonal Cellular Automata ($\mathcal{HCA}$) are 2D $\mathcal{CA}$ whose cells are of the form of a hexagonal. Morita et al.[6] introduced this type of cellular automaton and they called it $\mathcal{HCA}$. Image processing are excess more important compared with serial algorithms [9]. $\mathcal{CA}$ are widely used by researchers in the domain of image processing. So, $\mathcal{CA}$ can be used as a parallel method for any image processing task [3].

The paper is organized as follows.. In this second section, the concept used in the paper are formally defined. In this third section, the neighbor structure of 2D $\mathcal{HCA}$ is explained. In this forth section,rule matrix of $\mathcal{HCA}$ is studied. In this fifth section, we dispute about a few application in the field of image analysis using $\mathcal{HCA}$.

2. PRELIMINARIES

Definition 2.1. [5]A Null Boundary $\mathcal{CA}$ is the one in which the extreme cells are connected to logic zero state.

Definition 2.2. [7] **Uniform** $\mathcal{CA}$: The same rule applied to all the cells.

Definition 2.3. [1] **Hybrid** $\mathcal{CA}$: The different rules have to implement the different cells.

Definition 2.4. [2] **Cellular Automata**:($\mathcal{CA}$): $\mathcal{CA}$ is defined as a quadruplets $\mathcal{M} = \{d, \mathbf{Q}, \mathbf{N}, f\}$
* $d \in \mathbb{Z}_+$ is the dimension of the $\mathcal{CA}$.
* $\mathbf{Q} = \{1, 2, ..., p\}$ is a countable set of states.

2010 *Mathematics Subject Classification.* 37B15, 14G50, 68U10.
Key words and phrases. Cellular automata; Matrix algebra; Image analysis.
Received: November 15, 2020. Accepted: December 18, 2020. Published: December 31, 2020.
*Corresponding author.

* $\mathbf{N} = (\vec{n_1}, \vec{n_2}, ..., \vec{n_m})$ is the neighbor vector
* $f : \mathbf{Q}^m \to \mathbf{Q}$ is the local rule. f given the new states of a cell from the old neighbors states of the cells.

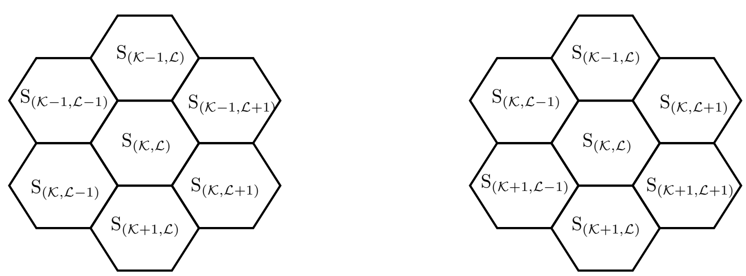
A mapping $\mathbb{C} : \mathbb{Z}^d \to \mathbf{Q}$. $\mathbb{C}^t$ is denote the time t, the cell move to next state at time t+1.

$\mathbb{C}^{t+1}(\vec{n}) = f(\mathbb{C}^t(\vec{n_1}), \mathbb{C}^t(\vec{n_2}), ..., \mathbb{C}^t(\vec{n_m}))$
now we consider f is a local rule of linear function
$\mathbb{C}^{t+1}(\vec{n}) = \lambda_1\mathbb{C}^t(\vec{n_1}) + \lambda_2\mathbb{C}^t(\vec{n_2}) + ... + \lambda_m\mathbb{C}^t(\vec{n_m}))$
λ_i is the co-efficient for neighborhood.

In [4] the state of the cell $(\mathcal{K}, \mathcal{L})$ at time t is denoted by $\mathrm{S}^{(t)}_{(\mathcal{K},\mathcal{L})}$. The state of the cell $(\mathcal{K}, \mathcal{L})$ at time (t+1) is denoted by $\mathrm{S}^{(t+1)}_{(\mathcal{K},\mathcal{L})} = \mathrm{R}^{(t)}_{(\mathcal{K},\mathcal{L})}$.
The rule matrix T_R that changes set of states of $\mathcal{CA}$ from (t) to (t+1) such that
$[\mathrm{S}]_{1\times mn}.(T_R)_{mn\times mn} = [\mathrm{R}]_{mn\times 1}$,
where
$([\mathrm{R}]_{mn\times 1}) = (\mathrm{S}^{(t+1)}_{11}, \mathrm{S}^{(t+1)}_{12}, \cdots, \mathrm{S}^{(t+1)}_{1n}, \cdots, \mathrm{S}^{(t+1)}_{m1}, \cdots, \mathrm{S}^{(t+1)}_{mn})$
$=(\mathrm{R}^{(t)}_{11}, \mathrm{R}^{(t)}_{12}, \cdots, \mathrm{R}^{(t)}_{1n}, \cdots, \mathrm{R}^{(t)}_{m1}, \cdots, \mathrm{R}^{(t)}_{mn})$.

3. The Neighborhood Structure of 2D $\mathcal{HCA}$

In this section, we show the neighborhood structure $\mathcal{HCA}$ over the field $\mathbb{Z}_2$ under the null boundary.



a. $\mathcal{L}$ is even positive integer b. $\mathcal{L}$ is odd positive integer

Figure 1. Two configuration of the $\mathcal{HCA}$.

In Figure 1, we show the $\mathcal{HCA}$ which comprises 6 cells surrounding the center cell $\mathrm{S}_{(\mathcal{K},\mathcal{L})}$ time t . The state of $\mathrm{S}_{(\mathcal{K},\mathcal{L})}$ at time $(t + 1)$ is a function $f : \mathbb{Z}_2^6 \to \mathbb{Z}_2$ defined as follows.

If $\mathcal{L}$ is an even integer figure 1.a, then we have,
$\mathrm{S}_{(\mathcal{K}-1,\mathcal{L})} + \mathrm{S}_{(\mathcal{K}-1,\mathcal{L}+1)} + \mathrm{S}_{(\mathcal{K},\mathcal{L}+1)} + \mathrm{S}_{(\mathcal{K}+1,\mathcal{L})} + \mathrm{S}_{(\mathcal{K},\mathcal{L}-1)} + \mathrm{S}_{(\mathcal{K}-1,\mathcal{L}-1)}$ ...(1)

If $\mathcal{L}$ is an odd integer figure 1.b, then we have,
$\mathrm{S}_{(\mathcal{K}-1,\mathcal{L})} + \mathrm{S}_{(\mathcal{K},\mathcal{L}+1)} + \mathrm{S}_{(\mathcal{K}+1,\mathcal{L}+1)} + \mathrm{S}_{(\mathcal{K}+1,\mathcal{L})} + \mathrm{S}_{(\mathcal{K}+1,\mathcal{L}-1)} + \mathrm{S}_{(\mathcal{K},\mathcal{L}-1)}$ ...(2)

4. RULE MATRIX OF THE $\mathcal{HCA}$ WITH NULL BOUNDARY

In this section, we discuss with the rule matrix of 2D $\mathcal{HCA}$ with null boundary over the field $\mathbb{Z}_2$.
Case (i). We take n is even positive integer and the following theorem.

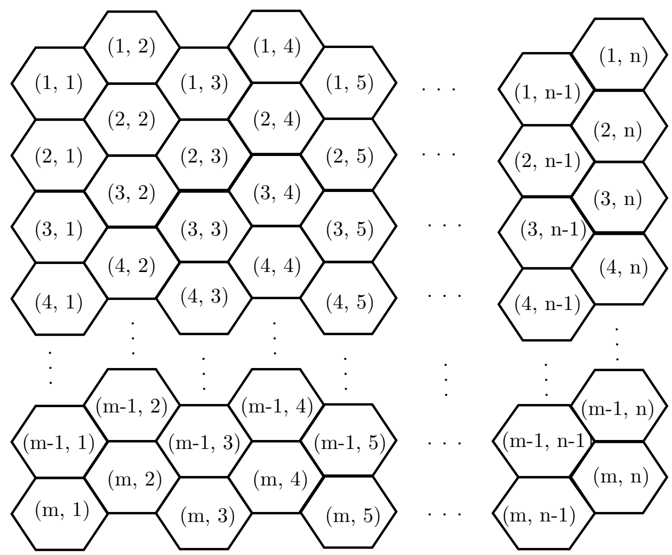


FIGURE 2. $\mathcal{HCA}$ of order $m \times n$
and n is even

Theorem 4.1. *Let $\mathcal{CA} = (d, S, N, f)$ be the $\mathcal{HCA}$. Let $m \geq 3$ and n be an even positive integer. We prove that there exist a rule matrix T_R^E from $\mathbb{Z}_2^{mn} \to \mathbb{Z}_2^{mn}$ corresponding to the 2D $\mathcal{HCA}$ which takes from configuration the state $\mathbb{C}^t$ of order $m \times n$ to the $(t+1)^{th}$ state $\mathbb{C}^{(t+1)}$ is given by,*

$$T_R^E = \begin{pmatrix} \mathcal{A}^E & \mathcal{B}^E & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \mathcal{C}^E & \mathcal{A}^E & \mathcal{B}^E & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \mathcal{C}^E & \mathcal{A}^E & \mathcal{B}^E & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \mathcal{C}^E & \mathcal{A}^E & \mathcal{B}^E & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & \mathcal{C}^E & \mathcal{A}^E & \mathcal{B}^E \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \mathcal{C}^E & \mathcal{A}^E \end{pmatrix}_{(mn \times mn)}$$

Where each sub matrix,

$$\mathcal{A}^E = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}_{(n \times n)}$$

$$\mathcal{B}^E = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}_{(n\times n)}$$

$$\mathcal{C}^E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix}_{(n\times n)} \qquad \textit{and}$$

0 is the zero matrix

Proof. Let $(\mathrm{S}_{(\mathcal{K},\mathcal{L})})T_R = \mathrm{R}_{(\mathcal{K},\mathcal{L})}$. $\mathrm{R}_{(\mathcal{K},\mathcal{L})} = \mathrm{S}^{(t+1)}_{(\mathcal{K},\mathcal{L})}$ is a equal to the linear combination of the neighbors in the following equation (1) and (2). The co-efficient of $\mathrm{S}_{\mathcal{K}\mathcal{L}} = 0$ if $\mathcal{K} \le 0$ *or* $\mathcal{L} \le 0$. By using the local rule of the $\mathcal{CA}$ we have obtain the following,

$\mathrm{R}_{(1,1)} = \mathrm{S}_{(1,2)} + \mathrm{S}_{(2,2)} + \mathrm{S}_{(2,1)}$

$\mathrm{R}_{(1,\mathcal{L})} = \mathrm{S}_{(1,\mathcal{L}+1)} + \mathrm{S}_{(2,\mathcal{L})} + \mathrm{S}_{(1,\mathcal{L}-1)},$ if $\mathcal{L}$ is even and $2 \ge \mathcal{L} > (n-1)$

$\mathrm{R}_{(1,\mathcal{L})} = \mathrm{S}_{(1,\mathcal{L}+1)} + \mathrm{S}_{(2,\mathcal{L}+1)} + \mathrm{S}_{(2,\mathcal{L})} + \mathrm{S}_{(2,\mathcal{L}-1)} + \mathrm{S}_{(1,\mathcal{L}-1)},$ if $\mathcal{L}$ is odd and $3 \ge \mathcal{L} \ge (n-1)$

$\mathrm{R}_{(1,n)} = \mathrm{S}_{(2,n)} + \mathrm{S}_{(1,n-1)}$

$\mathrm{R}_{(\mathcal{K},1)} = \mathrm{S}_{(\mathcal{K}-1,1)} + \mathrm{S}_{(\mathcal{K},2)} + \mathrm{S}_{(\mathcal{K}+1,2)} + \mathrm{S}_{(\mathcal{K}+1,1)}$, if $2 \ge \mathcal{K} \ge (n-1)$

$\mathrm{R}_{(\mathcal{K},\mathcal{L})} = \mathrm{S}_{(\mathcal{K}-1,\mathcal{L})} + \mathrm{S}_{(\mathcal{K}-1,\mathcal{L}+1)} + \mathrm{S}_{(\mathcal{K},\mathcal{L}+1)} + \mathrm{S}_{(\mathcal{K}+1,\mathcal{L})} + \mathrm{S}_{(\mathcal{K},\mathcal{L}-1)} + \mathrm{S}_{(\mathcal{K}-1,\mathcal{L}-1)},$ if $\mathcal{L}$ is even and $2 \ge \mathcal{L} > (n-1)$

$\mathrm{R}_{(\mathcal{K},\mathcal{L})} = \mathrm{S}_{(\mathcal{K}-1,\mathcal{L})} + \mathrm{S}_{(\mathcal{K},\mathcal{L}+1)} + \mathrm{S}_{(\mathcal{K}+1,\mathcal{L}+1)} + \mathrm{S}_{(\mathcal{K}+1,\mathcal{L})} + \mathrm{S}_{(\mathcal{K}+1,\mathcal{L}-1)} + \mathrm{S}_{(\mathcal{K},\mathcal{L}-1)},$ if $\mathcal{L}$ is odd and $3 \ge \mathcal{L} \ge (n-1)$

$\mathrm{R}_{(\mathcal{K},n)} = \mathrm{S}_{(\mathcal{K}+1,n)} + \mathrm{S}_{(\mathcal{K},n-1)} + \mathrm{S}_{(\mathcal{K}-1,n-1)} + \mathrm{S}_{(\mathcal{K}-1,n)}$

$\mathrm{R}_{(m,1)} = \mathrm{S}_{(m-1,1)} + \mathrm{S}_{(m,2)}$

$\mathrm{R}_{(m,\mathcal{L})} = \mathrm{S}_{(m-1,\mathcal{L})} + \mathrm{S}_{(m-1,\mathcal{L}+1)} + \mathrm{S}_{(m,\mathcal{L}+1)} + \mathrm{S}_{(m,\mathcal{L}-1)} + \mathrm{S}_{(m-1,\mathcal{L}-1)},$ if $\mathcal{L}$ is even and $2 \ge \mathcal{L} > (n-1)$

$\mathrm{R}_{(m,\mathcal{L})} = \mathrm{S}_{(m,\mathcal{L}-1)} + \mathrm{S}_{(m-1,\mathcal{L})} + \mathrm{S}_{(m,\mathcal{L}+1)},$ if $\mathcal{L}$ is odd and $2 \ge \mathcal{L} \ge (n-1)$

$\mathrm{R}_{(m,n)} = \mathrm{S}_{(m,n-1)} + \mathrm{S}_{(m-1,n-1)} + \mathrm{S}_{(m-1,n)}$

finally, we get the rule matrix of even case. $\square$

Case (ii). We take n is odd positive integer and the following theorem.

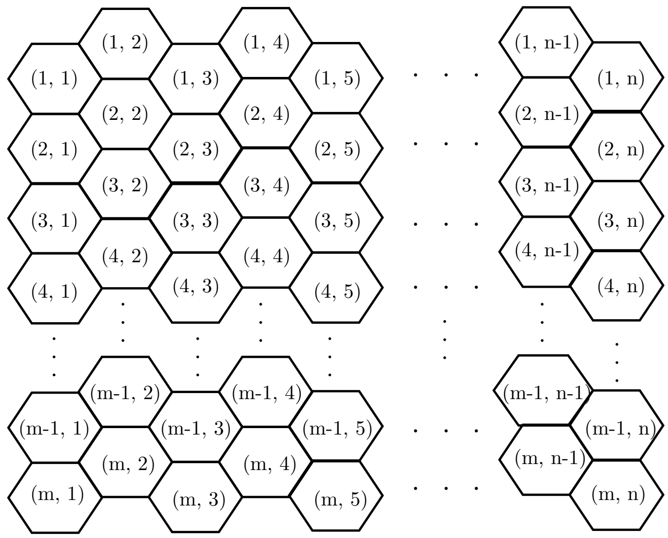


FIGURE 3. $\mathcal{HCA}$ of order $m \times n$ and n is odd

Theorem 4.2. *Let $\mathcal{CA} = (d, S, N, f)$ be the $\mathcal{HCA}$. Let $m \geq 3$ and n be an odd positive integer. We prove that there exist a rule matrix T_R^O from $\mathbb{Z}_2^{mn} \to \mathbb{Z}_2^{mn}$ corresponding to the 2D $\mathcal{HCA}$ which takes from configuration the state $\mathbb{C}^t$ of order $m \times n$ to the $(t+1)^{th}$ state $\mathbb{C}^{(t+1)}$ is given by,*

$$T_R^O = \begin{pmatrix} \mathcal{A}^O & \mathcal{B}^O & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \mathcal{C}^O & \mathcal{A}^O & \mathcal{B}^O & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \mathcal{C}^O & \mathcal{A}^O & \mathcal{B}^O & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \mathcal{C}^O & \mathcal{A}^O & \mathcal{B}^O & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & \mathcal{C}^O & \mathcal{A}^O & \mathcal{B}^O \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \mathcal{C}^O & \mathcal{A}^O \end{pmatrix}_{(mn \times mn)}$$

Where each sub matrix,

$$\mathcal{A}^O = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}_{(n \times n)}$$

$$\mathcal{B}^O = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix}_{(n\times n)}$$

$$\mathcal{C}^O = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}_{(n\times n)} \quad \textit{and}$$

0 is the zero matrix

Proof. The proof of theorem 4.2 can receive the following alike the same steps as in the proof of theorem 4.1. □

5. APPLICATION OF $\mathcal{HCA}$ IMAGE ANALYSIS

Two dimensional $\mathcal{HCA}$ algorithm are widely used in image processing as its shape is like to an image. In this section, we discuss the basic image processing of transition, zooming, boundary, and thinning.

5.1. **Transition.** Transition is very important to the part of image processing. The image moving from all the direction is using the transition. In this paper, we have applied seven basic of 2D $\mathcal{HCA}$ rules. The directions for the rules is indicated in the table below.

Table 1. Translation of images using basic 2D $\mathcal{HCA}$ rules.

Rules	Direction of translation of images
1	center
2	top
4	right-top
8	right-bottom
16	bottom
32	left-bottom
64	left-top

This rules using the hexagonal grid.
Translation of an image using 2D $\mathcal{CA}$ rules represented the following figure.

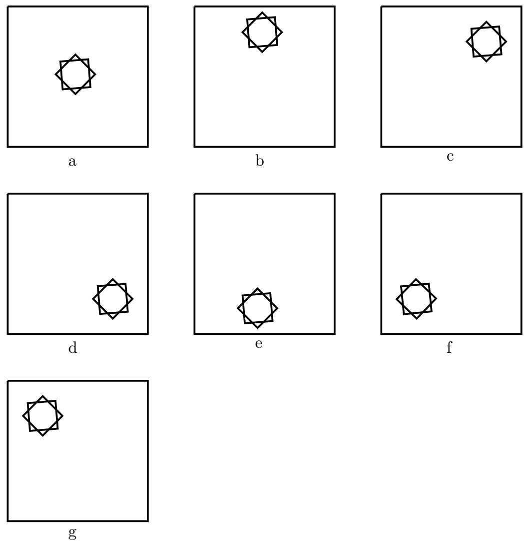
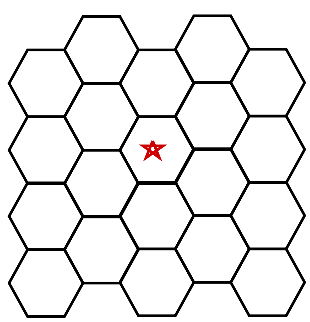


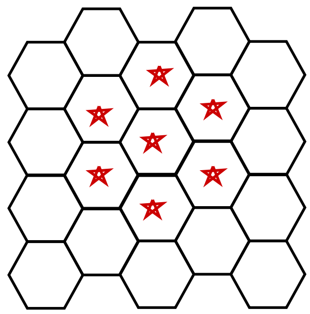
FIGURE 4. Translation of an image using 2D rules (a)Center Images (b) Top (c)right-top (d) right-bottom (e) bottom (f)left-bottom (g) left-top

5.2. **Zooming.** In zooming there are two operations, zooming in and zooming out. The following example of zooming demonstrates that row or column using different uniform and hybrid rules.
This example of zooming in

Example 5.1. Let us consider a 4×5 2D H with the starting configuration as shown in figure

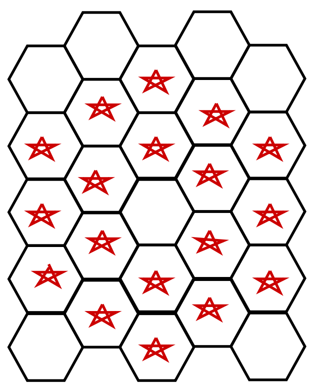


After running the $\mathcal{HCA}$ rules mentions in table foe every cells the resulting configuration is shown in below

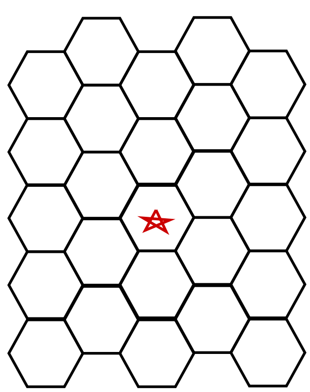


This example of zooming out.

Example 5.2. Let us consider a 5×5 2D $\mathcal{HCA}$ with the initial configuration as shown in figure

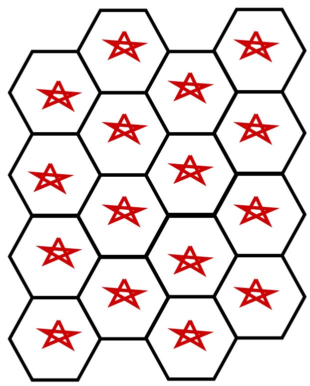


Every cell in the 1^{st} row, use the rule 1, rule 2, rule 4, rule 8, rule 16, rule 32 and rule 64.
Every cell in the 2^{nd} row, use the rule 1 and rule 16.
Every cell in the 3^{rd} row, use the rule 1 and rule 16.
Every cell in the 4^{th} row, use the rule 1 and rule 2 except the 3^{rd} cell that only use the rule 1 and 4.
In 5^{th} row 1^{st} cell we use rule 2 and rule 4. 2^{nd} cell we use rule 2, rule 4, rule 8 and rule 64. 3^{rd} cell we use rule 1, rule 2, rule 4 and rule 64. 4^{th} cell we use rule 2, rule 4, rule 32 and rule 64. 5^{th} cell using rule 2 and rule 64.
After applying this hybrid rules, the resulting configuration is shown in figure.

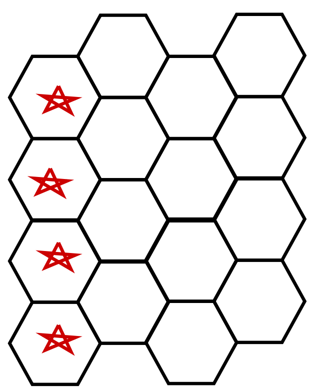


5.3. **Thinning.** Thinning is an important procedure in image analysis. The following example of thinning demonstrates that row or column can be thinned using different uniform and hybrid rules.

Example 5.3. Let us consider a 4×4 2D $\mathcal{HCA}$ containing all 1's with the starting configuration as shown in figure



Every cell in the 1^{st} column, use the rule 4, rule 8 and rule 16 if j is odd
Every cell in the 2^{nd} column, use the rule 1 and rule 8 if j is even
Every cell in the 3^{rd} column, use the rule 1 and rule 4 if j is odd
Every cell in the 4^{th} column, use the rule 1 and rule 32 if j is even



6. Conclusions

In this paper we have defined $\mathcal{HCA}$ local rule over the field $\mathbb{Z}_2$. The rule matrix associated to the 2D $\mathcal{HCA}$ has been obtained. We apply some important image process tasks such transition, zooming and thinning using 2D $\mathcal{HCA}$.

7. Acknowledgements

The authors would like to thank from the anonymous reviewers for carefully reading of the manuscript and giving useful comments, which will help us to improve the paper.

REFERENCES

[1] P. Anghelescu, S. Ionita and E. Safron. FPGA Implementation of Hybrid Additive Programmable Cellular Automata. In: Eight International Conference on Hybrid Intelligent Systems. IEEE, 2008.
[2] Bin yang, Chao wang and Aiyun Xiang. Reversibility of general 1D linear cellular automata over the binary field $\mathbb{Z}_2$ under null boundary conditions. Information Sciences, 324, 2015, p 23-31.
[3] C. Chang, Y. Zhang and Y. Gdong. Cellular Automata for Edge Detection of Images. IEEE proceedings on Machine Learning and Cybernetics., 2004, p 26-29.
[4] Irfan Siap, Hasan Akin and Selman Uguz. Structure and reversibility of 2D hexagonal cellular automata. Computers and Mathematics with Applications, 62, 2011, p 4161-4169.
[5] A. R. Khan, S. Mitra and P. Sarkar. VLSI Architecture of a cellular automata machine. Computers Math. Applic. 33(5), 1997, p 79-94.
[6] K. Morita, M. Margenstern, and K. Imai. Universality of reversible hexagonal cellular automata. RAIRO-Theoretical informatics and Applications, 33(6), 1999, p 535-550.
[7] S. Nandi, B. K. Kar and P. Pal Chaudhuri. Theory and applications of cellular automata in cryptography. IEEE Trans. Comput, 43(12), 1994, p 1346-1357.
[8] J. V. Neumann. Theory of Self-Reproducing Automata. University of Illinois Press, 1966.
[9] S. Sadeghi, A. Rezvanian and E. Kamrani. An efficient method for impulse noise reduction from images using fuzzy cellular automata. International Journal of Elec. and Comm, 2012, p 772-779.
[10] S. Ulam. Some Ideas and Prospects in Biomathematics. Annual Review of Biophysics and Bioengineering, 1963, p 277-292.
[11] S. Wolfram. Computation Theory of Cellular Automata. Commun. Math. Phys, 1984, p 15-57.

M. RAJASEKAR
ENGINEERING MATHEMATICS, ANNAMALAI UNIVERSITY, ANNAMALAINAGER, CHIDAMBARAM, 608 002, INDIA.
Email address: mdivraj1962@gmail.com

R. ANBU
DEPARTMENT OF MATHEMATICS, ANNAMALAI UNIVERSITY, ANNAMALAINAGER, CHIDAMBARAM, 608 002, INDIA.
Email address: anburaja2291@gmail.com

ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 3, Number 4 (2020), 303-308
ISSN: 2582-0818

SOMEWHAT PAIRWISE FUZZY e-IRRESOLUTE MAPPINGS

A. SWAMINATHAN* AND M. SANKARI

ABSTRACT. The concepts of somewhat pairwise fuzzy e-irresolute mapping and somewhat pairwise fuzzy irresolute e-open mapping is introduced. Also some properties and comparisions of those mappings are given.

1. INTRODUCTION

The concepts of fuzzy sets introduced by Zadeh[7]. Chang [2] studied the notion of fuzzy topology in 1968. Seenivasan and Kamala [3] introduced the concept of fuzzy e-continous functions in fuzzy topological spaces. The concepts of somewhat fuzzy e-continuous functions and somewhat fuzzy e-open functions are introduced and studied by Swaminathan in [6]. The purpose of this paper is to introduce and study the concepts of somewhat pairwise fuzzy e-irresolute mappings and somewhat pairwise fuzzy irresolute e-open mappings on a fuzzy bitopological spaces and also we discuss some of their properties.

A fuzzy subset A of a space X is called fuzzy regular open [1] (resp. fuzzy regular closed) if $A =$Int(Cl(A)) (resp.$A =$Cl(Int(A))). Now Cl(A) and Int(A) are defined as follows:Cl$(A) = \bigwedge \{U : U \geq A,\ U$ is fuzzy closed in $X\}$ and
Int$(A) = \bigvee \{U \leq A,\ U$ is fuzzy open in $X\}$. The fuzzy δ-interior of a fuzzy subset A of X is the union of all fuzzy regular open sets contained in A. A fuzzy subset A is called fuzzy δ-open [4] if $A = \text{Int}_\delta(A)$. The complement of fuzzy δ-open set is called fuzzy δ-closed (i.e, $A = \text{Cl}_\delta(A)$).

A fuzzy subset A of a space X is called fuzzy e-open[3] if $A \leq$cl(int$_\delta A$)$\vee$int(cl$_\delta A$) and fuzzy e-closed set if $A \geq$cl(int$_\delta A$)$\wedge$int(cl$_\delta A$). Throughout this paper X and Y stand for $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(Y, \mathcal{F}_1, \mathcal{F}_2)$ respectively.

Definition 1.1. *A mapping $f : X \to Y$ is called fuzzy e-continuous* [3] *if $f^{-1}(V)$ is a fuzzy e-open set on X for any fuzzy open set V on Y.*

Definition 1.2. *A mapping $f : X \to Y$ is called somewhat fuzzy e-continuous*[6] *if there exists a fuzzy e-open set $U \neq 0_X$ on X such that $U \leq f^{-1}(V) \neq 0_X$ for any fuzzy open set V on Y.*

2010 *Mathematics Subject Classification.* 54A40, 03E72.
Key words and phrases. Somewhat pairwise fuzzy e-irresolute mapping, somewhat pairwise fuzzy irresolute e-open mapping.
Received: November 25, 2020. Accepted: December 19, 2020. Published: December 31, 2020.
*Corresponding author.

Definition 1.3. *A mapping $f : X \to Y$ is called somewhat fuzzy* e*-open*[6] *if there exists a fuzzy* e*-open set $V \neq 0_Y$ on Y such that $V \leq f(U) \neq 0_Y$ for any fuzzy open set U on X.*

Definition 1.4. *A fuzzy set U on a fuzzy topological space X is called fuzzy* e*-dense*[6] *if there exists no fuzzy* e*-closed set V in X such that $U < V < 1$.*

2. SOMEWHAT PAIRWISE FUZZY e-IRRESOLUTE MAPPINGS

In this section we introduce a somewhat pairwise fuzzy e-irresolute mapping and compared few results.

Definition 2.1. *A mapping $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{F}_1, \mathcal{F}_2)$ is called pairwise fuzzy e-continuous if $f^{-1}(V)$ is a $\mathcal{T}_1$-fuzzy e-open or $\mathcal{T}_2$-fuzzy e-open set on $(X, \mathcal{T}_1, \mathcal{T}_2)$ for any $\mathcal{F}_1$-fuzzy open or $\mathcal{F}_2$-fuzzy open set V on $(Y, \mathcal{F}_1, \mathcal{F}_2)$.*

Definition 2.2. *A mapping $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{F}_1, \mathcal{F}_2)$ is called pairwise fuzzy e-irresolute if $f^{-1}(V)$ is a $\mathcal{T}_1$-fuzzy e-open or $\mathcal{T}_2$-fuzzy e-open set on $(X, \mathcal{T}_1, \mathcal{T}_2)$ for any $\mathcal{F}_1$-fuzzy e-open or $\mathcal{F}_2$-fuzzy e-open set V on $(Y, \mathcal{F}_1, \mathcal{F}_2)$.*

Definition 2.3. *A mapping $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{F}_1, \mathcal{F}_2)$ is called somewhat pairwise fuzzy e-continuous if there exists a $\mathcal{T}_1$-fuzzy e-open or $\mathcal{T}_2$-fuzzy e-open set $U \neq 0_X$ on $(X, \mathcal{T}_1, \mathcal{T}_2)$ such that $U \leq f^{-1}(V) \neq 0_X$ for any $\mathcal{F}_1$-fuzzy open or $\mathcal{F}_2$-fuzzy open set $V \neq 0_Y$ on $(Y, \mathcal{F}_1, \mathcal{F}_2)$.*

Definition 2.4. *A mapping $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{F}_1, \mathcal{F}_2)$ is called somewhat pairwise fuzzy e-irresolute if there exists a $\mathcal{T}_1$-fuzzy e-open or $\mathcal{T}_2$-fuzzy e-open set $U \neq 0_X$ on $(X, \mathcal{T}_1, \mathcal{T}_2)$ such that $U \leq f^{-1}(V) \neq 0_X$ for any $\mathcal{F}_1$-fuzzy e-open or $\mathcal{F}_2$-fuzzy e-open set $V \neq 0_Y$ on $(Y, \mathcal{F}_1, \mathcal{F}_2)$.*

Remark. *From the above definitions, it is observed that the following reverse implications are false:*
(i)Every pairwise fuzzy e-continuous mapping is a somewhat pairwise fuzzy e-continuous mapping.
(ii)Every somewhat pairwise fuzzy e-irresolute mapping is a somewhat pairwise fuzzy e-continuous mapping.
(iii)Every pairwise fuzzy e-irresolute mapping is a somewhat pairwise fuzzy e-irresolute mapping.

Example 2.5. Let M_1,M_2,M_3,M_4 and M_5 be fuzzy sets on $X = Y = \{x, y, z\}$. Then $M_1 = \frac{0.3}{a} + \frac{0.4}{b} + \frac{0.5}{c}$, $M_2 = \frac{0.6}{a} + \frac{0.5}{b} + \frac{0.5}{c}$, $M_3 = \frac{0.7}{a} + \frac{0.6}{b} + \frac{0.4}{c}$, $M_4 = \frac{0.3}{x} + \frac{0.4}{y} + \frac{0.4}{z}$ and $M_5 = \frac{0.6}{x} + \frac{0.5}{y} + \frac{0.4}{z}$ are defined as follows:Consider $\mathcal{T}_1 = \{0_X, M_1, M_2, M_4, M_5, 1_X\}$,$\mathcal{T}_2 = \{0_X, M_1, M_2, M_4, 1_X\}$,$\mathcal{F}_1 = \{0_X, M_1, M_1^c, M_2, M_3, M_4, M_5, 1_X\}$ and
$\mathcal{F}_2 = \{0_X, M_1, M_5, 1_X\}$. Then $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(Y, \mathcal{F}_1, \mathcal{F}_2)$ are fuzzy bitopologies and $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{F}_1, \mathcal{F}_2)$ be an identity mapping. Then we have $M_1 \leq f^{-1}(M_1) = M_1$, $M_1 \leq f^{-1}(M_1^c) = M_1^c$,$M_1 \leq f^{-1}(M_2) = M_2$,$M_4 \leq f^{-1}(M_3) = M_3$,$M_4 \leq f^{-1}(M_4) = M_4$ and $M_4 \leq f^{-1}(M_5) = M_5$. Since M_1, M_2 and M_4 are $\mathcal{T}_1$-fuzzy e-open set on $(X, \mathcal{T}_1, \mathcal{T}_2)$, f is somewhat pairwise fuzzy e-continuous. But $f^{-1}(M_3) = M_3$ is not a $\mathcal{T}_1$-fuzzy e-open or $\mathcal{T}_2$-fuzzy e-open set on $(X, \mathcal{T}_1, \mathcal{T}_2)$. Hence f is not a pairwise fuzzy e-continuous mapping.

Example 2.6. *Let $\mu_1(x)$, $\mu_2(x)$ and $\mu_3(x)$ be fuzzy sets on $I = [0, 1]$ defined as follows:*

$$\mu_1(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2} \\ 2x - 1, & \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$\mu_2(x) = \begin{cases} 1, & 0 \leq x \leq \frac{1}{2} \\ -4x+2, & \frac{1}{2} \leq x \leq \frac{3}{4} \\ 0, & \frac{3}{4} \leq x \leq 1 \end{cases}$$

$$\mu_3(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{4} \\ \frac{1}{3}(4x-1), & \frac{1}{4} \leq x \leq 1 \end{cases}$$

Let $\mathcal{T}_1 = \{0, \mu_1, \mu_2, \mu_1 \vee \mu_2, 1\}$ and $\mathcal{T}_2 = \{0, \mu_1 \vee \mu_2, 1\}$, $\mathcal{T}_3 = \{0, \mu_2^{'}, 1\}$ and $\mathcal{T}_4 = \{0, \mu_1^{'}, 1\}$ be a fuzzy topologies on I. Let $f : (I, \mathcal{T}_1, \mathcal{T}_2) \to (I, \mathcal{F}_1, \mathcal{F}_2)$ be a function defined by $f(x) = \frac{x}{2}$ for each $x \in I$. We can see that for fuzzy e-open sets $\mu_1^{'}$ and $\mu_2^{'}$ on $(I, \mathcal{F}_1, \mathcal{F}_2)$, $f^{-1}(\mu_1^{'}) = 1_X$,$\mu_1 \leq f^{-1}(\mu_2^{'}) = \mu_1$. Since μ_1 is a fuzzy e-open set on $(I, \mathcal{T}_1, \mathcal{T}_2)$. Therefore f is somewhat pairwise fuzzy e-continuous mapping. Consider a fuzzy open set μ_3 which is fuzzy e-open set on $(I, \mathcal{T}_1, \mathcal{T}_2)$, $f^{-1}(\mu_3) = \mu_3 f(\frac{x}{2}) = \alpha(x) =$ $\begin{cases} 0, & 0 \leq x \leq \frac{1}{2} \\ \frac{1}{3}(2x-1), & \frac{1}{2} \leq x \leq 1 \end{cases}$ for each $x \in I$. But there is no non-zero fuzzy e-open set smaller than $f^{-1}(\mu_3(x)) = \alpha(x)$ on $(I, \mathcal{T}_1, \mathcal{T}_2)$. Hence f is not somewhat pairwise fuzzy e-irresolute mapping.

Example 2.7. *In Example 2.5, for an $\mathcal{F}_1$-fuzzy e-open sets on $(Y, \mathcal{F}_1, \mathcal{F}_2)$, $M_4 \leq f^{-1}(M_1) = M_1$, $M_4 \leq f^{-1}(M_1^c) = M_1^c$,$M_4 \leq f^{-1}(M_2) = M_2$,$M_4 \leq f^{-1}(M_3) = M_3$,$M_4 \leq f^{-1}(M_4) = M_4$ and $M_4 \leq f^{-1}(M_5) = M_5$. Since M_4 is a $\mathcal{T}_1$-fuzzy e-open set on $(X, \mathcal{T}_1, \mathcal{T}_2)$, f is somewhat pairwise fuzzy e-continuous But $f^{-1}(M_3) = M_3$ is not a $\mathcal{T}_1$-fuzzy or $\mathcal{T}_2$-fuzzy e-open set on $(X, \mathcal{T}_1, \mathcal{T}_2)$. Hence f is not a pairwise fuzzy e-irresolute mapping.*

Definition 2.8. *A fuzzy set U on a fuzzy bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is called pairwise e-dense fuzzy set if there exists no $\mathcal{T}_1$-fuzzy e-closed or $\mathcal{T}_2$-fuzzy e-closed set V in $(X, \mathcal{T}_1, \mathcal{T}_2)$ such that $U < V < 1$.*

Theorem 2.1. Let $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{F}_1, \mathcal{F}_2)$ be a mapping. Then the following are equivalent:
(1) f is somewhat pairwise fuzzy e-irresolute .
(2) If V is an $\mathcal{F}_1$-fuzzy e-closed or $\mathcal{F}_2$-fuzzy e-closed set of $(Y, \mathcal{F}_1, \mathcal{F}_2)$ such that $f^{-1}(V) \neq 1_X$, then there exists a $\mathcal{T}_1$-fuzzy e-closed or $\mathcal{T}_2$-fuzzy e-closed set $U \neq 1_X$ of $(X, \mathcal{T}_1, \mathcal{T}_2)$ such that $f^{-1}(V) \leq U$.
(3) If U is a pairwise e-dense fuzzy set on $(X, \mathcal{T}_1, \mathcal{T}_2)$, then $f(U)$ is a pairwise e-dense fuzzy set on $(Y, \mathcal{F}_1, \mathcal{F}_2)$.

Proof. (1)$\Rightarrow$ (2): Let V be an $\mathcal{F}_1$-fuzzy e-closed or $\mathcal{F}_2$-fuzzy e-closed set on $(Y, \mathcal{F}_1, \mathcal{F}_2)$ such that $f^{-1}(V) \neq 1_X$. Then V^c is an $\mathcal{F}_1$-fuzzy e-open or $\mathcal{F}_2$-fuzzy e-open set on $(Y, \mathcal{F}_1, \mathcal{F}_2)$ and $f^{-1}(V^c) = (f^{-1}(V))^c \neq 0_X$. Since f is somewhat pairwise e-irresolute , there exists a $\mathcal{T}_1$-fuzzy e-open or $\mathcal{T}_2$-fuzzy e-open set $U^c \neq 0_X$ on $(X, \mathcal{T}_1, \mathcal{T}_2)$ such that $U^c \leq f^{-1}(V^c)$. Hence there exists $\mathcal{T}_1$-fuzzy e-closed or $\mathcal{T}_2$-fuzzy e-closed set $U \neq 0_X$ on $(X, \mathcal{T}_1, \mathcal{T}_2)$ such that $f^{-1}(V) = 1 - f^{-1}(V^c) \leq 1 - U^c = U$.

(2) $\Rightarrow$ (3): Let U be a pairwise e-dense fuzzy set on $(X, \mathcal{T}_1, \mathcal{T}_2)$ and suppose $f(U)$ is not pairwise e-dense fuzzy set on $(Y, \mathcal{F}_1, \mathcal{F}_2)$. Then there exists an $\mathcal{F}_1$-fuzzy e-closed or $\mathcal{F}_2$-fuzzy e-closed set V on $(Y, \mathcal{F}_1, \mathcal{F}_2)$ such that $f(U) < V < 1$. Since $V < 1$ and $f^{-1}(V) \neq 1_X$, there exists a $\mathcal{T}_1$-fuzzy e-closed or $\mathcal{T}_2$-fuzzy e-closed set $D \neq 1_X$ such that $U \leq f^{-1}(f(U)) < f^{-1}(V) \leq D$. This contradicts to the assumption that U is a

pairwise e-dense fuzzy set on $(X, \mathcal{T}_1, \mathcal{T}_2)$. Hence $f(U)$ is a pairwise e-dense fuzzy set on $(Y, \mathcal{F}_1, \mathcal{F}_2)$.

(3) $\Rightarrow$ (1): Let $V \neq 0_Y$ be an $\mathcal{F}_1$-fuzzy e-open or $\mathcal{F}_2$-fuzzy e-open set on $(Y, \mathcal{F}_1, \mathcal{F}_2)$ and let $f^{-1}(V) \neq 0_X$. Suppose that there exists no $\mathcal{T}_1$-fuzzy e-open or $\mathcal{T}_2$-fuzzy e-open set $U \neq 0_X$ on $(X, \mathcal{T}_1, \mathcal{T}_2)$ such that $U \leq f^{-1}(V)$. Then $(f^{-1}(V))^c$ is a $\mathcal{T}_1$-fuzzy set or $\mathcal{T}_2$-fuzzy set on $(X, \mathcal{T}_1, \mathcal{T}_2)$ such that there is no $\mathcal{T}_1$-fuzzy e-closed or $\mathcal{T}_2$-fuzzy e-closed set D on $(X, \mathcal{T}_1, \mathcal{T}_2)$ with $(f^{-1}(V))^c < D < 1$. In fact, if there exists a $\mathcal{T}_1$-fuzzy e-open or $\mathcal{T}_2$-fuzzy e-open set D^c such that $D^c \leq f^{-1}(V)$, then it is a contradiction. So $(f^{-1}(V))^c$ is a pairwise e-dense fuzzy set on $(X, \mathcal{T}_1, \mathcal{T}_2)$. Then $f((f^{-1}(V))^c)$ is a pairwise e-dense fuzzy set on $(Y, \mathcal{F}_1, \mathcal{F}_2)$. But $f((f^{-1}(V))^c) = f(f^{-1}(V^c)) \neq V^c < 1$. This is a contradiction to the fact that $f((f^{-1}(V))^c)$ is pairwise e-dense fuzzy set on $(Y, \mathcal{F}_1, \mathcal{F}_2)$. Hence there exists a $\mathcal{T}_1$-fuzzy e-open or $\mathcal{T}_2$-fuzzy e-open set $U \neq 0_X$ on $(X, \mathcal{T}_1, \mathcal{T}_2)$ such that $U \leq f^{-1}(V)$. Consequently, f is somewhat pairwise fuzzy e-irresolute . $\square$

Theorem 2.2. Let $(X_1, \mathcal{T}_1, \mathcal{T}_2), (X_2, \mathcal{G}_1, \mathcal{G}_2), (Y_1, \mathcal{F}_1, \mathcal{F}_2), (Y_2, \mathcal{K}_1, \mathcal{K}_2)$ be fuzzy bitopological spaces. Let $(X_1, \mathcal{T}_1, \mathcal{T}_2)$ be product related to $(X_2, \mathcal{G}_1, \mathcal{G}_2)$ and let $(Y_1, \mathcal{F}_1, \mathcal{F}_2)$ be product related to $(Y_2, \mathcal{K}_1, \mathcal{K}_2)$. If $f_1 : (X_1, \mathcal{T}_1, \mathcal{T}_2) \to (Y_1, \mathcal{F}_1, \mathcal{F}_2)$ and $f_2 : (X_2, \mathcal{G}_1, \mathcal{G}_2) \to (Y_2, \mathcal{K}_1, \mathcal{K}_2)$ is a somewhat pairwise fuzzy e-irresolute mappings, then the product $f_1 \times f_2 : (X_1, \mathcal{T}_1, \mathcal{T}_2) \times (X_2, \mathcal{G}_1, \mathcal{G}_2) \to (Y_1, \mathcal{F}_1, \mathcal{F}_2) \times (Y_2, \mathcal{K}_1, \mathcal{K}_2)$ is also somewhat pairwise fuzzy e-irresolute.

Proof. Let $C = \bigvee_{i,j} (U_i \times V_j)$ be $\mathcal{F}_i$-fuzzy e-open or $\mathcal{K}_j$-fuzzy e-open set on $(Y_1, \mathcal{F}_1, \mathcal{F}_2) \times (Y_2, \mathcal{K}_1, \mathcal{K}_2)$ where $U_i \neq 0_{Y_1}$ is $\mathcal{F}_i$-fuzzy e-open set and $V_j \neq 0_{Y_2}$ is $\mathcal{K}_j$-fuzzy e-open set on $(Y_1, \mathcal{F}_1, \mathcal{F}_2)$ and $(Y_2, \mathcal{K}_1, \mathcal{K}_2)$ respectively. Then $(f_1 \times f_2)^{-1}(C) = \bigvee_{i,j} (f_1^{-1}(U_i) \times f_2^{-1}(V_j))$. Since f_1 is somewhat pairwise fuzzy e-irresolute , there exists a $\mathcal{T}_1$-fuzzy e-open or $\mathcal{T}_2$-fuzzy e-open set $D_i \neq 0_{X_1}$ such that $D_i \leq f_1^{-1}(U_i) \neq 0_{X_1}$. And, since f_2 is somewhat pairwise fuzzy e-irresolute , there exists a $\mathcal{G}_1$-fuzzy e-open or $\mathcal{G}_2$-fuzzy e-open set $A_j \neq 0_{X_2}$ such that $A_j \leq f_2^{-1}(V_j) \neq 0_{X_2}$. Now $D_i \times A_j \leq f_1^{-1}(U_i) \times f_2^{-1}(V_j) = (f_1 \times f_2)^{-1}(U_i \times V_j)$ and $D_i \times A_j \neq 0_{X_1 \times X_2}$ is a D_i-fuzzy e-open or V_j-fuzzy e-open set on $(X_1, \mathcal{T}_1, \mathcal{T}_2) \times (X_2, \mathcal{G}_1, \mathcal{G}_2)$. Hence $D_i \times A_j$ is a $\mathcal{T}_i$-fuzzy e-open or $\mathcal{G}_j$-fuzzy e-open set on $(X_1, \mathcal{T}_1, \mathcal{T}_2) \times (X_2, \mathcal{G}_1, \mathcal{G}_2)$ such that $\bigvee_{i,j} (D_i \times A_j) \leq \bigvee_{i,j} (f_1^{-1}(U_i) \times f_2^{-1}(V_j)) = (f_1 \times f_2)^{-1}(\bigvee_{i,j} (U_i \times V_j)) = (f_1 \times f_2)^{-1}(C) \neq 0_{X_1 \times X_2}$. Therefore, $f_1 \times f_2$ is somewhat pairwise fuzzy e-irresolute . $\square$

Theorem 2.3. Let $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{F}_1, \mathcal{F}_2)$ be a mapping. If the graph $g : (X, \mathcal{T}_1, \mathcal{T}_2) \to (X, \mathcal{T}_1, \mathcal{T}_2) \times (Y, \mathcal{F}_1, \mathcal{F}_2)$ of f is a somewhat pairwise fuzzy e-irresolute mapping, then f is also somewhat pairwise fuzzy e-irresolute .

Proof. Let V be an $\mathcal{F}_1$-fuzzy e-open or $\mathcal{F}_2$-fuzzy e-open set on $(Y, \mathcal{F}_1, \mathcal{F}_2)$. Then $f^{-1}(V) = 1 \wedge f^{-1}(V) = g^{-1}(1 \times V)$. Since g is somewhat pairwise fuzzy e-irresolute and $1 \times V$ is a $\mathcal{T}_i$-fuzzy e-open or $\mathcal{F}_j$-fuzzy e-open set on $(X, \mathcal{T}_1, \mathcal{T}_2) \times (Y, \mathcal{F}_1, \mathcal{F}_2)$, there exists a $\mathcal{T}_1$-fuzzy e-open or $\mathcal{T}_2$-fuzzy e-open set $U \neq 0_X$ on $(X, \mathcal{T}_1, \mathcal{T}_2)$ such that $U \leq g^{-1}(1 \times V) = f^{-1}(V) \neq 0_X$. Therefore, f is somewhat pairwise fuzzy e-irresolute . $\square$

3. Somewhat pairwise fuzzy irresolute e-open mappings

In this section, we introduce a somewhat pairwise fuzzy irresolute e-open mapping and we characterize a somewhat pairwise fuzzy irresolute e-open mapping.

Definition 3.1. *A mapping $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{F}_1, \mathcal{F}_2)$ is called pairwise fuzzy e-open if $f(U)$ is an $\mathcal{F}_1$-fuzzy e-open or $\mathcal{F}_2$-fuzzy e-open set on $(Y, \mathcal{F}_1, \mathcal{F}_2)$ for any $\mathcal{T}_1$-fuzzy open or $\mathcal{T}_2$-fuzzy open set U on $(X, \mathcal{T}_1, \mathcal{T}_2)$.*

Definition 3.2. *A mapping $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{F}_1, \mathcal{F}_2)$ is called pairwise fuzzy irresolute e-open if $f(U)$ is an $\mathcal{F}_1$-fuzzy e-open or $\mathcal{F}_2$-fuzzy e-open set on $(Y, \mathcal{F}_1, \mathcal{F}_2)$ for any $\mathcal{T}_1$-fuzzy e-open or $\mathcal{T}_2$-fuzzy e-open set U on $(X, \mathcal{T}_1, \mathcal{T}_2)$.*

Definition 3.3. *A mapping $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{F}_1, \mathcal{F}_2)$ is called somewhat pairwise fuzzy e-open if there exists an $\mathcal{F}_1$-fuzzy e-open or $\mathcal{F}_2$-fuzzy e-open set $V \neq 0_Y$ on $(Y, \mathcal{F}_1, \mathcal{F}_2)$ such that $V \leq f(U) \neq 0_Y$ for any $\mathcal{T}_1$-fuzzy open or $\mathcal{T}_2$-fuzzy open set U on $(X, \mathcal{T}_1, \mathcal{T}_2)$.*

Definition 3.4. *A mapping $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{F}_1, \mathcal{F}_2)$ is called somewhat pairwise fuzzy irresolute e-open if there exists an $\mathcal{F}_1$-fuzzy e-open or $\mathcal{F}_2$-fuzzy e-open set $V \neq 0_Y$ on $(Y, \mathcal{F}_1, \mathcal{F}_2)$ such that $V \leq f(U) \neq 0_Y$ for any $\mathcal{T}_1$-fuzzy e-open or $\mathcal{T}_2$-fuzzy e-open set $U \neq 0_X$ on $(X, \mathcal{T}_1, \mathcal{T}_2)$.*

Theorem 3.1. Let $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{F}_1, \mathcal{F}_2)$ be a bijection. Then the following are equivalent:
(1) f is somewhat pairwise fuzzy irresolute e-open.
(2) If U is a $\mathcal{T}_1$-fuzzy e-closed or $\mathcal{T}_2$-fuzzy e-closed set on $(X, \mathcal{T}_1, \mathcal{T}_2)$ such that $f(U) \neq 1_Y$, then there exists an $\mathcal{F}_1$-fuzzy e-closed or $\mathcal{F}_2$-fuzzy e-closed set $V \neq 1_Y$ on $(Y, \mathcal{F}_1, \mathcal{F}_2)$ such that $f(U) < V$.

Proof. (1) $\Rightarrow$ (2): Let U be a $\mathcal{T}_1$-fuzzy e-closed or $\mathcal{T}_2$-fuzzy e-closed set on $(X, \mathcal{T}_1, \mathcal{T}_2)$ such that $f(U) \neq 1_Y$. Since f is bijective and U^c is a $\mathcal{T}_1$-fuzzy e-open or $\mathcal{T}_2$-fuzzy e-open set on $(X, \mathcal{T}_1, \mathcal{T}_2)$, $f(U^c) = (f(U))^c \neq 0_Y$. And, since f is somewhat pairwise fuzzy irresolute e-open mapping, there exists an $\mathcal{F}_1$-fuzzy e-open or $\mathcal{F}_2$-fuzzy e-open set $D \neq 0_Y$ on $(Y, \mathcal{F}_1, \mathcal{F}_2)$ such that $D < f(U^c) = (f(U))^c$. Consequently, $f(U) < D^c = V \neq 1_Y$ and V is an $\mathcal{F}_1$-fuzzy e-closed or $\mathcal{F}_2$-fuzzy e-closed set on $(Y, \mathcal{F}_1, \mathcal{F}_2)$.

(2) $\Rightarrow$ (1): Let U be a $\mathcal{T}_1$-fuzzy e-open or $\mathcal{T}_2$-fuzzy e-open set on $(X, \mathcal{T}_1, \mathcal{T}_2)$ such that $f(U) \neq 0_Y$. Then U^c is a $\mathcal{T}_1$-fuzzy e-closed or $\mathcal{T}_2$-fuzzy e-closed set on $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $f(U^c) \neq 1_Y$. Hence there exists an $\mathcal{F}_1$-fuzzy e-closed or $\mathcal{F}_2$-fuzzy e-closed set $V \neq 1_Y$ on $(Y, \mathcal{F}_1, \mathcal{F}_2)$ such that $f(U^c) < V$. Since f is bijective, $f(U^c) = (f(U))^c < V$. Hence $V^c < f(U)$ and $V^c \neq 0_X$ is an $\mathcal{F}_1$-fuzzy e-open or $\mathcal{F}_2$-fuzzy e-open set on $(Y, \mathcal{F}_1, \mathcal{F}_2)$. Therefore, f is somewhat pairwise fuzzy irresolute e-open.

□

Theorem 3.2. Let $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{F}_1, \mathcal{F}_2)$ be a surjection. Then the following are equivalent:
(1) f is somewhat pairwise fuzzy irresolute e-open.
(2) If V is a pairwise e-dense fuzzy set on $(Y, \mathcal{F}_1, \mathcal{F}_2)$, then $f^{-1}(V)$ is a pairwise e-dense fuzzy set on $(X, \mathcal{T}_1, \mathcal{T}_2)$.

Proof. (1) $\Rightarrow$ (2): Let V be a pairwise e-dense fuzzy set on $(Y, \mathcal{F}_1, \mathcal{F}_2)$. Suppose $f^{-1}(V)$ is not pairwise e-dense fuzzy set on $(X, \mathcal{T}_1, \mathcal{T}_2)$. Then there exists a $\mathcal{T}_1$-fuzzy e-closed or $\mathcal{T}_2$-fuzzy e-closed set U on $(X, \mathcal{T}_1, \mathcal{T}_2)$ such that $f^{-1}(V) < U < 1$. Since f is somewhat pairwise fuzzy irreolute e-open and U^c is a $\mathcal{T}_1$-fuzzy e-open or $\mathcal{T}_2$-fuzzy e-open set on $(X, \mathcal{T}_1, \mathcal{T}_2)$, there exists an $\mathcal{F}_1$-fuzzy e-open or $\mathcal{F}_2$-fuzzy e-open set $D \neq 0_Y$ on $(Y, \mathcal{F}_1, \mathcal{F}_2)$ such that $D \leq f(IntU^c) \leq f(U^c)$. Since f is surjective, $D \leq f(U^c) < f(f^{-1}(V^c)) = V^c$. Thus there exists an $\mathcal{F}_1$-fuzzy e-closed or $\mathcal{F}_2$-fuzzy e-closed set D^c

on $(Y, \mathcal{F}_1, \mathcal{F}_2)$ such that $V < D^c < 1$. This is a contradiction. Hence $f^{-1}(V)$ is pairwise e-dense fuzzy set on $(X, \mathcal{T}_1, \mathcal{T}_2)$.

(2) $\Rightarrow$ (1): Let U be a $\mathcal{T}_1$-fuzzy open or $\mathcal{T}_2$-fuzzy open set on $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $f(U) \neq 0_Y$. Suppose there exists no $\mathcal{F}_1$-fuzzy e-open or $\mathcal{F}_2$-fuzzy e-open set $V \neq 0_Y$ on $(Y, \mathcal{F}_1, \mathcal{F}_2)$ such that $V \leq f(U)$. Then $(f(U))^c$ is an $\mathcal{F}_1$-fuzzy set or $\mathcal{F}_2$-fuzzy set D on $(Y, \mathcal{F}_1, \mathcal{F}_2)$ such that there exists no $\mathcal{F}_1$-fuzzy e-closed or $\mathcal{F}_2$-fuzzy e-closed set D on $(Y, \mathcal{F}_1, \mathcal{F}_2)$ with $(f(U))^c < D < 1$. This means that $(f(U))^c$ is pairwise e-dense fuzzy set on $(Y, \mathcal{F}_1, \mathcal{F}_2)$. Thus $f^{-1}((f(U))^c)$ is pairwise e-dense fuzzy set on $(X, \mathcal{T}_1, \mathcal{T}_2)$. But $f^{-1}((f(U))^c) = (f^{-1}(f(U)))^c \leq U^c < 1$. This is a contradiction to the fact that $f^{-1}(f(V))^c$ is pairwise e-dense fuzzy set on $(X, \mathcal{T}_1, \mathcal{T}_2)$. Hence there exists an $\mathcal{F}_1$-fuzzy e-open or $\mathcal{F}_1$-fuzzy e-open set $V \neq 0_Y$ on $(Y, \mathcal{F}_1, \mathcal{F}_2)$ such that $V \leq f(U)$. Therefore, f is somewhat pairwise fuzzy irresolute e-open. $\square$

4. Conclusions and/or Discussions

Even though the concept of somewhat fuzzy continuous functions are not at all fuzzy continuous functions , it has some interseting stuff to develope further. In this aspect the somewhat fuzzy δ-irresolute continuous and somewhat fuzzy e-irresolute continuous mappings were investigated in [5] and [6] respectively. From these papers, we have developed and studied this research article in fuzzy bitopology as somewhat pairwise fuzzy e-irresolute mappings;this paper only concerned with somewhat pairwise fuzzy e-irresolute functions with some interesting properties.

5. Acknowledgements

The authors are thankful to the referees for their suggestions and commands to develop this manuscript.

References

[1] K.K. Azad, On fuzzy semi-continuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl., 82(1981), 14-32.
[2] C.L Chang, Fuzzy topological spaces, J. Math. Anal. Appl., 24(1968), 182-190.
[3] V. Seenivasan and K. Kamala, Fuzzy e-continuity and fuzzy e-open sets, Ann. of Fuzzy Math. and Inform., 81(2014), 141-148.
[4] Supriti Saha, Fuzzy δ-continuous mappings, J. Math. Anal. Appl., 126(1987), 130-142.
[5] A. Swaminathan and K. Balasubramaniyan, Somewhat fuzzy δ-irresolute continuous mappings, Ann. of Fuzzy Math. and Inform., 12(1)(2016), 121-128.
[6] A. Swaminathan, Somewhat fuzzy e-irresolute continuous mappings, The J. Fuzzy Math., 27(2)(2019), 271-279.
[7] L. A. Zadeh, Fuzzy sets, Information and control 8 (1965), 338-353.

A. Swaminathan
Department of Mathematics, Government Arts College(Autonomous), Kumbakonam, Tamil Nadu-612 002, India.
Email address: asnathanway@gmail.com

M. Sankari
Department of Mathematics, Lekshmipuram College of Arts and Science, Neyyoor, Kanyakumari, Tamil Nadu-629 802, India.
Email address: sankarisaravanan1968@gmail.com

ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 3, Number 4 (2020), 309-316

DIRECT PRODUCT OF ANTI $\mathcal{N}$-H-IDEALS IN BCK-ALGEBRAS

ANAS AL-MASARWAH

ABSTRACT. In this paper, we introduce the notion of anti $\mathcal{N}$-H-ideals of a BCK-algebra. Then, the notion of the direct product of two anti $\mathcal{N}$-H-ideals by using minimum operation is introduced, and some related properties are studied. Ordinary H-ideals are linked with anti $\mathcal{N}$-H-ideals by means of an anti $\mathcal{N}$-s-level set of the direct product of two $\mathcal{N}$-structures.

1. INTRODUCTION

The study of BCK-algebras was introduced by Imai and Iséki [12] in 1966. BCK-algebras have been applied to many branches of mathematics, such as functional analysis, group theory, topology, probability theory. Since Imai and Iséki [12] introduced the concepts of ideals in BCK-algebras, many types of ideals in BCK-algebras have occurred, for instance, H-ideals, closed ideals, implicative ideals, positive implicative ideals, and so on.

A crisp set C in a universe X is a function $\lambda_C : X \to \{0,1\}$ yielding the value 0 for elements excluded from the set C and the value 1 for elements belonging to the set C. As a generalization of crisp sets, Zadeh [20] introduced the degree of positive membership in 1965 and defined the concept of fuzzy set theory. This concept was applied to a BCK-algebra by Xi [19]. Jun et al. [14] presented a new function which is called negative-valued function, and developed $\mathcal{N}$-structure as one of the hybrid models of fuzzy set. They applied the idea of $\mathcal{N}$-structure in BCK-algebras and proposed $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals [14]. In [13], Jun established the definition of doubt fuzzy subalgebras and ideals in BCK-algebras. Al-Masarwah et al. [10] introduced the notions of doubt $\mathcal{N}$-subalgebras and ideals in BCK-algebras, and discussed several properties. After that, many Hybrid models of fuzzy sets were applied in BCK-algebras and other algebraic structures [5, 6, 8, 7, 9, 4, 17, 18, 2, 1, 3, 16, 15].

In this paper, we discuss an $\mathcal{N}$-structure with an application to BCK-algebras. We introduce the notion of anti $\mathcal{N}$-H-ideals in a BCK-algebra. Also, we considered the structure of a BCK-algebra and defined the direct product of two anti $\mathcal{N}$-H-ideals. We present some interesting results about direct product of two anti $\mathcal{N}$-H-ideals of a BCK-algebra.

2010 *Mathematics Subject Classification.* 06F35, 03G25, 03B52, 03B05.
Key words and phrases. BCK-algebra; H-ideal; $\mathcal{N}$-structure; Anti $\mathcal{N}$-ideal; Anti $\mathcal{N}$-H-ideal.
Received: November 30, 2020. Accepted: December 24, 2020. Published: December 31, 2020.

Finally, we proved that the direct product of two anti $\mathcal{N}$-structures becomes an anti $\mathcal{N}$-H-ideal if and only if for any $s \in [-1, 0]$, an anti $\mathcal{N}$-s-level set is an H-ideal of a BCK-algebra $X \times Y$.

2. Preliminaries

In this section, we include some basic definitions and preliminary facts about a BCK-algebra which are essential for our results. By a BCK-algebra, we mean an algebra $(X; *, 0)$ of type (2, 0) satisfying the following axioms for all $x, y, z \in X$:
(I) $((x * y) * (x * z)) * (z * y) = 0,$
(II) $(x * (x * y)) * y = 0,$
(III) $x * x = 0,$
(IV) $0 * x = 0,$
(V) $x * y = 0$ and $y * x = 0$ imply $x = y$.

Any BCK-algebra X satisfies the following axioms for all $x, y, z \in X$:
(I1) $x * 0 = x,$
(I2) $(x * y) * z = (x * z) * y,$
(I3) $x * y \leq x,$
(I4) $(x * y) * z \leq (x * z) * (y * z),$
(I5) $x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x.$

A partial ordering $\leq$ on a BCK-algebra X can be defined by $x \leq y$ if and only if $x * y = 0$. A non-empty subset K of a BCK/BCI-algebra X is called:

(1) A subalgebra of a BCK-algebra X [12] if $x * y \in K, \forall x, y \in X,$
(2) An ideal of a BCK-algebra X [12] if $\forall x, y \in X,$
   - $0 \in K,$
   - $x * y \in K$ and $y \in K$ imply $x \in K.$
(3) An H-ideal of a BCK-algebra X [11] if $\forall x, y, z \in X,$
   - $0 \in K,$
   - $((x * y) * z) \in K$ and $y \in K$ imply $x * z \in K.$

Definition 2.1. [21] A fuzzy set $A = \{(x, \mu_A(x)) \mid x \in X\}$ in a BCK-algebra X is called an anti (a doubt) fuzzy H-ideal of X if

(1) $\mu_A(0) \leq \mu_A(x),$
(2) $\mu_A(x * z) \leq \max\{\mu_A(x * (y * z)), \mu_A(y)\}$, for all $x, y, z \in X.$

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to the interval $[-1, 0]$. We say that, an element of $\mathcal{F}(X, [-1, 0])$ is a negative-valued function from X to [-1, 0] (briefly, $\mathcal{N}$-function on X). By an $\mathcal{N}$-structure we mean an ordered pair (X, ϕ), where φ is an $\mathcal{N}$-function on X. In what follows, φ is an $\mathcal{N}$-function on X unless otherwise specified.

In [14], Jun et al. introduced the concepts of $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals in a BCK-algebra as follows:

Definition 2.2. An $\mathcal{N}$-structure (X, φ) is called an $\mathcal{N}$-subalgebra of X if for all $x, y \in X$:

$$\varphi(x * y) \leq \max\{\varphi(x), \varphi(y)\}.$$

Definition 2.3. An $\mathcal{N}$-structure (X, φ) is called an $\mathcal{N}$-ideal of X if for all $x, y \in X$:

(1) $\varphi(0) \leq \varphi(x),$
(2) $\varphi(x) \leq \max\{\varphi(x * y), \varphi(y)\}.$

3. Direct product of anti $\mathcal{N}$-H-Ideals

In this section, we introduce the concept of an anti $\mathcal{N}$-H-ideal. Then, we give the definition of the direct product of two $\mathcal{N}$-H-ideals of two BCK-algebras X and Y, and we provide some of its properties.

In what follows, X and Y are BCK-algebras, so we use $(X \times Y; *, (0,0))$ to denote a BCK-algebra unless otherwise specified. For the sake of brevity, we call $X \times Y$ a BCK-algebra.

Definition 3.1. An $\mathcal{N}$-structure (X, φ) is called an anti $\mathcal{N}$-H-ideal of X if it satisfies the following conditions for all $x, y, z \in X$:

(1) $\varphi(0) \geq \varphi(x)$,
(2) $\varphi(x * z) \geq \min\{\varphi(x * (y * z)), \varphi(y)\}$.

Example 3.2. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let (X, φ) be an $\mathcal{N}$-structure in which φ is given by:

$$\varphi(x) = \begin{cases} -0.1, & \text{if } x = 0 \\ -0.2, & \text{if } x = a,b,c. \end{cases}$$

Then by routine calculation, we know that (X, φ) is an anti $\mathcal{N}$-H-ideal of X.

Definition 3.3. Let $(X, *_X, 0_X)$ and $(Y, *_Y, 0_Y)$ be two BCK-algebras. The direct product of X and Y is defined to be the set $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$. In $X \times Y$ we define the product $*_{X \times Y}$ as follows:

$$(x, y) *_{X \times Y} (u, v) = (x *_X u, y *_Y v) \text{ for all } (x, y), (u, v) \in X \times Y.$$

One can easily verify that the direct product of two BCK-algebras is again a BCK-algebra. Now, we write the following definition.

Definition 3.4. Let X be a BCK-algebra and let (X, φ) and (X, ϕ) be two anti $\mathcal{N}$-H-ideals of X. The direct product of (X, φ) and (X, ϕ) is defined by $(X \times X, \varphi \times \phi)$, where $\varphi \times \phi : X \times X \to [-1, 0]$ is given by

$$(\varphi \times \phi)(x, y) = \min\{\varphi(x), \phi(y)\}$$

for all $(x, y) \in X \times X$.

In the following, we extend the above definition to the direct product of anti $\mathcal{N}$-H-ideals of any BCK-algebras X and Y.

Definition 3.5. Let X and Y be two BCK-algebras and let (X, φ) and (Y, ϕ) be two anti $\mathcal{N}$-H-ideals of X and Y, respectively. Then, the direct product of (X, φ) and (Y, ϕ) is defined by $(X \times Y, \varphi \times \phi)$, where $\varphi \times \phi : X \times Y \to [-1, 0]$ is given by

$$(\varphi \times \phi)(x, y) = \min\{\varphi(x), \phi(y)\}$$

for all $(x, y) \in X \times Y$.

Definition 3.6. An $\mathcal{N}$-structure $(X \times Y, \varphi \times \phi)$ of a BCK-algebra $X \times Y$ is called an anti $\mathcal{N}$-H-ideal of $X \times Y$ if it satisfies the following conditions for all $(x, y), (u, v), (w, z) \in X \times Y$:

(1) $(\varphi\times\phi)(0,0)\geq(\varphi\times\phi)(x,y)$,
(2) $(\varphi\times\phi)((x,y)*(w,z))\geq\min\{(\varphi\times\phi)((x,y)*((u,v)*(w,z))),(\varphi\times\phi)(u,v)\}$.

Example 3.7. Consider a BCK-algebra $X=\{0,a,b,c\}$ and an anti $\mathcal{N}$-H-ideal (X,φ) of X which are given in Example 3.2. Define an anti $\mathcal{N}$-H-ideal (X,ϕ) in X as follows:

$$\phi(x)=\begin{cases}-0.3, & \text{if } x=0\\ -0.4, & \text{if } x=a,b,c.\end{cases}$$

Consider $(X\times X,\varphi\times\phi)$, where $(\varphi\times\phi)(x,y)=\min\{\varphi(x),\phi(y)\}$ is defined as:

$$(\varphi\times\phi)(x,y)=\begin{cases}-0.3, \text{ if } & (x,y)=(0,0),(a,0),(b,0),(c,0)\\ -0.4, & \text{otherwise.}\end{cases}$$

By routine calculations, we know that $(X\times X,\varphi\times\phi)$ is an anti $\mathcal{N}$-H-ideal of $X\times X$.

Theorem 3.1. *Let (X,φ) and (Y,ϕ) be two anti $\mathcal{N}$-H-ideals of BCK-algebras X and Y, respectively. Then, $(X\times Y,\varphi\times\phi)$ is an anti $\mathcal{N}$-H-ideal of $X\times Y$.*

Proof. For any $(x,y)\in X\times Y$, we have

$$(\varphi\times\phi)(0,0)=\min\{\varphi(0),\phi(0)\}\geq\min\{\varphi(x),\phi(y)\}=(\varphi\times\phi)(x,y).$$

Now, for any $(x,y),(u,v),(w,z)\in X\times Y$, we have

$$\begin{aligned}(\varphi\times\phi)((x,y)*(w,z))&=(\varphi\times\phi)(x*w,y*z)\\&=\min\{\varphi(x*w),\phi(y*z)\}\\&\geq\min\{\min\{\varphi(x*(u*w)),\varphi(u)\},\min\{\phi(y*(v*z)),\phi(v)\}\}\\&=\min\{\min\{\varphi(x*(u*w)),\phi(y*(v*z))\},\min\{\varphi(u),\phi(v)\}\}\\&=\min\{(\varphi\times\phi)((x*(u*w)),(y*(v*z))),(\varphi\times\phi)(u,v)\}\\&=\min\{(\varphi\times\phi)((x,y)*((u,v)*(w,z))),(\varphi\times\phi)(u,v)\}.\end{aligned}$$

Hence, $(X\times Y,\varphi\times\phi)$ is an anti $\mathcal{N}$-H-ideal of $X\times Y$. □

Proposition 3.2. *Let $(X\times Y,\varphi\times\phi)$ be an anti $\mathcal{N}$-H-ideal of $X\times Y$. If $(x,y)\leq(u,v)$, then $(\varphi\times\phi)(x,y)\geq(\varphi\times\phi)(u,v)$ for all $(x,y),(u,v)\in X\times Y$.*

Proof. Let $(x,y),(u,v)\in X\times Y$ such that $(x,y)\leq(u,v)$. Then, $(x,y)*(u,v)=(0,0)$. Now,

$$\begin{aligned}(\varphi\times\phi)(x,y)&=(\varphi\times\phi)((x,y)*(0,0))\\&\geq\min\{(\varphi\times\phi)((x,y)*((u,v)*(0,0))),(\varphi\times\phi)(u,v)\}\\&=\min\{(\varphi\times\phi)((x,y)*(u,v)),(\varphi\times\phi)(u,v)\}\\&=\min\{(\varphi\times\phi)(0,0),(\varphi\times\phi)(u,v)\}=(\varphi\times\phi)(u,v).\end{aligned}$$

Therefore, $(\varphi\times\phi)(x,y)\geq(\varphi\times\phi)(u,v)$ for all $(x,y),(u,v)\in X\times Y$. □

Proposition 3.3. *Let $(X\times Y,\varphi\times\phi)$ be an anti $\mathcal{N}$-H-ideal of $X\times Y$ such that*

$$(\varphi\times\phi)((x,y)*(u,v))\geq(\varphi\times\phi)(u,v)$$

for all $(x,y),(u,v)\in X\times Y$. Then, $(X\times Y,\varphi\times\phi)$ is an $\mathcal{N}$-constant.

Proof. Note that in a BCK-algebra $X\times Y$, $(x,y)*(0,0)=(x,y)$ for all $(x,y)\in X\times Y$, and by using the assumption, we have

$$(\varphi\times\phi)(x,y)=(\varphi\times\phi)((x,y)*(0,0))\geq(\varphi\times\phi)(0,0).$$

It follows from Definition 3.6, $(\varphi\times\phi)(x,y) = (\varphi\times\phi)(0,0)$ for all $(x,y),(u,v)\in X\times Y$. Therefore, $(X\times Y, \varphi\times\phi)$ is an $\mathcal{N}$-constant. $\square$

Proposition 3.4. *Let (X,φ) and (Y,ϕ) be two anti $\mathcal{N}$-H-ideals of X and Y, respectively. If $(X\times Y,\varphi\times\phi)$ is an anti $\mathcal{N}$-H-ideal of $X\times Y$, then $\varphi(0)\geq\phi(y)$ and $\phi(0)\geq\varphi(x)$ for all $x\in X, y\in Y$.*

Proof. Assume that $\varphi(0)<\phi(y)$ and $\phi(0)<\varphi(x)$ for some $x\in X, y\in Y$. Then,

$$\begin{aligned}(\varphi\times\phi)(x,y) &= \min\{\varphi(x),\phi(y)\}\\ &> \min\{\varphi(0),\phi(0)\}\\ &= (\varphi\times\phi)(0,0),\end{aligned}$$

which is a contradiction. Thus, $\varphi(0)\geq\phi(y)$ and $\phi(0)\geq\varphi(x)$ for all $x\in X, y\in Y$. $\square$

Theorem 3.5. *Let (X,φ) and (Y,ϕ) be two $\mathcal{N}$-structures of X and Y, respectively, such that $(X\times Y,\varphi\times\phi)$ is an anti $\mathcal{N}$-H-ideal of $X\times Y$. Then, either (X,φ) is an anti $\mathcal{N}$-H-ideal of X or (Y,ϕ) is an anti $\mathcal{N}$-H-ideal of Y.*

Proof. Since $(X\times Y,\varphi\times\phi)$ is an anti $\mathcal{N}$-H-ideal of $X\times Y$, then for all $(x,y),(u,v),(w,z)\in X\times Y$, we have

$$(\varphi\times\phi)((x,y)*(w,z))\geq\min\{(\varphi\times\phi)((x,y)*((u,v)*(w,z))),(\varphi\times\phi)(u,v)\}.$$

By putting $y=z=v=0$, we have

$$(\varphi\times\phi)((x,0)*(w,0))\geq\min\{(\varphi\times\phi)((x,0)*((u,0)*(w,0))),(\varphi\times\phi)(u,0)\}. \quad (3.1)$$

Also, we have

$$\begin{aligned}(\varphi\times\phi)((x,0)*(w,0)) &= (\varphi\times\phi)((x*w),(0*0))\\ &= \min\{\varphi(x*w),\phi(0*0)\}\\ &= \varphi(x*w) \qquad (3.2)\end{aligned}$$

and

$$\begin{aligned}(\varphi\times\phi)((x,0)*((u,0)*(w,0))) &= (\varphi\times\phi)((x,0)*((u*w),(0*0)))\\ &= (\varphi\times\phi)((x*(u*w)),(0*(0,0)))\\ &= \min\{\varphi(x*(u*w)),\phi(0*(0,0))\}\\ &= \varphi(x*(u*w)). \qquad (3.3)\end{aligned}$$

Again, by using Proposition 3.4, we have

$$(\varphi\times\phi)(u,0)=\min\{\varphi(u),\phi(0)\}=\varphi(u). \quad (3.4)$$

So, from (3.1), (3.2), (3.3) and (3.4) we get, $\varphi(x*w)\geq\min\{\varphi(x*(u,w)),\varphi(u)\}$. Hence, (X,φ) is an anti $\mathcal{N}$-H-ideal of X. $\square$

Proposition 3.6. *Let $(X\times Y,\varphi\times\phi)$ be an anti $\mathcal{N}$-H-ideal of a BCK-algebra $X\times Y$. Then,*

$$(\varphi\times\phi)((0,0)*((0,0)*(x,y)))\geq(\varphi\times\phi)(x,y)$$

for all $(x,y)\in X\times Y$.

Proof. Note that

$$\begin{aligned}
&(\varphi\times\phi)((0,0)*((0,0)*(x,y)))\\
&\qquad\geq \min\{(\varphi\times\phi)((0,0)*((x,y)*((0,0)*(x,y)))),(\varphi\times\phi)(x,y)\}\\
&\qquad= \min\{(\varphi\times\phi)((0,0)*((x,y)*(0,0))),(\varphi\times\phi)(x,y)\}\\
&\qquad= \min\{(\varphi\times\phi)((0,0)*(x,y)),(\varphi\times\phi)(x,y)\}\\
&\qquad= \min\{(\varphi\times\phi)(0,0),(\varphi\times\phi)(x,y)\}\\
&\qquad= (\varphi\times\phi)(x,y) \text{ for all } (x,y)\in X\times Y.
\end{aligned}$$

Therefore, $(\varphi\times\phi)((0,0)*((0,0)*(x,y)))\geq(\varphi\times\phi)(x,y)$ for all $(x,y)\in X\times Y$. □

Corollary 3.7. *Let $(X\times Y,\varphi\times\phi)$ be an anti $\mathcal{N}$-H-ideal of $X\times Y$. Then, the set*

$$D_{(\varphi\times\phi)}=\{(x,y)\in X\times Y\mid(\varphi\times\phi)(x,y)=(\varphi\times\phi)(0,0)\}$$

is an H-ideal of $X\times Y$.

Proof. Let $(X\times Y,\varphi\times\phi)$ be an anti $\mathcal{N}$-H-ideal of $X\times Y$. Obviously, $(0,0)\in D_{(\varphi\times\phi)}$. Let $(x,y),(u,v),(w,z)\in D_{(\varphi\times\phi)}$ be such that $(x,y)*((u,v)*(w,z)),(u,v)\in D_{(\varphi\times\phi)}$. Then,

$$(\varphi\times\phi)((x,y)*((u,v)*(w,z)))=(\varphi\times\phi)(0,0)=(\varphi\times\phi)(u,v).$$

Now,

$$\begin{aligned}
(\varphi\times\phi)((x,y)*(w,z)) &\geq \min\{(\varphi\times\phi)((x,y)*((u,v)*(w,z))),(\varphi\times\phi)(u,v)\}\\
&= (\varphi\times\phi)(0,0).
\end{aligned}$$

Again, since $(X\times Y,\varphi\times\phi)$ is an anti $\mathcal{N}$-H-ideal of $X\times Y$, so $(\varphi\times\phi)(0,0)\geq(\varphi\times\phi)((x,y)*(w,z))$. Therefore, $(\varphi\times\phi)(0,0)=(\varphi\times\phi)((x,y)*(w,z))$. It follows that $(x,y)*(w,z)\in D_{(\varphi\times\phi)}$ for all $(x,y),(u,v),(w,z)\in X\times Y$. Therefore, $D_{(\varphi\times\phi)}$ is an H-ideal of X. □

Definition 3.8. Let $(X\times Y,\varphi\times\phi)$ be an anti $\mathcal{N}$-H-ideal of $X\times Y$ and $s\in[-1,0]$. Then, an anti $\mathcal{N}$-s-level set of $(X\times Y,\varphi\times\phi)$ is as follows:

$$(\varphi\times\phi)_s=\{(x,y)\in X\times Y\mid(\varphi\times\phi)(x,y)\geq s\}.$$

Theorem 3.8. *Let $(X\times Y,\varphi\times\phi)$ be an $\mathcal{N}$-structure of $X\times Y$. Then, $(X\times Y,\varphi\times\phi)$ is an anti $\mathcal{N}$-H-ideal of $X\times Y$ if and only if $(\varphi\times\phi)_s\neq\emptyset$ is an H-ideal of $X\times Y$ for all $s\in[-1,0]$.*

Proof. Assume that $(X\times Y,\varphi\times\phi)$ is an anti $\mathcal{N}$-H-ideal of $X\times Y$ and $s\in[-1,0]$ such that $(\varphi\times\phi)_s\neq\emptyset$. Let $(c,d)\in(\varphi\times\phi)_s$. Then, we have $(\varphi\times\phi)(c,d)\geq s$. So we deduce that $(\varphi\times\phi)(0,0)\geq(\varphi\times\phi)(c,d)\geq s$. This shows that $(0,0)\in(\varphi\times\phi)_s$. Let $(x',y'),(u',v'),(w',z')\in X\times Y$ such that $(x',y')*((u',v')*(w',z'))\in(\varphi\times\phi)_s$ and $(u',v')\in(\varphi\times\phi)_s$. Then, $(\varphi\times\phi)((x',y')*((u',v')*(w',z')))\geq s$ and $(\varphi\times\phi)(u',v')\geq s$. Since $(X\times Y,\varphi\times\phi)$ is an anti $\mathcal{N}$-H-ideal of $X\times Y$, it follows that

$$\begin{aligned}
&(\varphi\times\phi)((x',y')*(w',z'))\\
&\qquad\geq \min\{(\varphi\times\phi)((x',y')*((u',v')*(w',z'))),(\varphi\times\phi)(u',v')\}\\
&\qquad\geq \min\{s,s\}=s,
\end{aligned}$$

so $(x',y')*(w',z')\in(\varphi\times\phi)_s$. Therefore, $(\varphi\times\phi)_s$ is an H-ideal of $X\times Y$.

Conversely, assume that $(\varphi \times \phi)_s \neq \emptyset$ is an H-ideal of $X \times Y$ for all $s \in [-1, 0]$. Let $(x, y) \in X \times Y$ be such that $(\varphi \times \phi)(0, 0) < (\varphi \times \phi)(x, y)$. By taking

$$s_o = \frac{1}{2}[(\varphi \times \phi)(0, 0) + (\varphi \times \phi)(x, y)],$$

we get $(\varphi \times \phi)(0, 0) < s_o < (\varphi \times \phi)(x, y)$. Therefore, $(0, 0) \notin (\varphi \times \phi)_{s_o}$. This is a contradiction. Hence, $(\varphi \times \phi)(0, 0) \geq (\varphi \times \phi)(x, y)$ for all $(x, y) \in X \times Y$. Again, we assume that $(x, y), (u, v), (w, z) \in X \times Y$ be such that

$$(\varphi \times \phi)((x, y) * (w, z)) < \min\{(\varphi \times \phi)((x, y) * ((u, v) * (w, z))), (\varphi \times \phi)(u, v)\}.$$

By taking

$$s_1 = \frac{1}{2}[(\varphi \times \phi)((x, y) * (w, z)) + \min\{(\varphi \times \phi)((x, y) * ((u, v) * (w, z))), (\varphi \times \phi)(u, v)\}],$$

we have

$$(\varphi \times \phi)((x, y) * (w, z)) < s_1 < \min\{(\varphi \times \phi)((x, y) * ((u, v) * (w, z))), (\varphi \times \phi)(u, v)\}.$$

This shows that, $(x, y) * ((u, v) * (w, z)) \in (\varphi \times \phi)_{s_1}$, $(u, v) \in (\varphi \times \phi)_{s_1}$, but $(x, y) * (w, z) \notin (\varphi \times \phi)_{s_1}$, this is a contradiction. Therefore,

$$(\varphi \times \phi)((x, y) * (w, z)) \geq \min\{(\varphi \times \phi)((x, y) * ((u, v) * (w, z))), (\varphi \times \phi)(u, v)\}$$

for all $(x, y), (u, v), (w, z) \in X \times Y$. Hence, $(X \times Y, \varphi \times \phi)$ is an anti $\mathcal{N}$-H-ideal of $X \times Y$. $\square$

4. CONCLUSIONS

In this work, we introduced the notion of anti $\mathcal{N}$-H-ideals in a BCK-algebra. Also, we considered the structure of a BCK-algebra and defined the direct product of two anti $\mathcal{N}$-H-ideals. We presented some interesting results about the direct product of two anti $\mathcal{N}$-H-ideals of a BCK-algebra. Finally, we proved that the direct product of two anti $\mathcal{N}$-structures becomes an anti $\mathcal{N}$-H-ideal if and only if for any $s \in [-1, 0]$, an anti $\mathcal{N}$-s-level cut set is an H-ideal of a BCK-algebra $X \times Y$.

5. ACKNOWLEDGEMENTS

The author thanks the editors of this journal for their support in designing this paper.

REFERENCES

[1] M. Abu Qamar and N. Hassan, Characterizations of group theory under Q-neutrosophic soft environment, Neutrosophic Sets and Systems, 27(1) (2019), 114-130.

[2] M. Abu Qamar and N. Hassan, On Q-neutrosophic subring, Journal of Physics: Conference Series, 1212(1)(2019), 012018.

[3] M. Abu Qamar, A.G. Ahmad and N. Hassan, An approach to Q-neutrosophic soft rings, AIMS Mathematics, 4(4)(2019), 1291-1306.

[4] A. Al-Masarwah and A.G. Ahmad, Doubt bipolar fuzzy subalgebras and ideals in BCK/BCI- algebras, Journal of Mathematical Analysis, 9(3)(2018), 9-27.

[5] A. Al-Masarwah and A.G. Ahmad, m-Polar fuzzy ideals of BCK/BCI-algebras, Journal of King Saud University-Science, 31(4)(2019), 1220-1226.

[6] A. Al-Masarwah and A.G. Ahmad, m-Polar (α, β)-fuzzy ideals in BCK/BCI-algebras, Symmetry, 11(1)(2019), 44. DOI:10.3390/sym11010044.

[7] A. Al-Masarwah and A.G. Ahmad, Novel concepts of doubt bipolar fuzzy H-ideals of BCK/BCI-algebras, International Journal of Innovative Computing, Information and Control, 14(6)(2018), 2025-2041.

[8] A. Al-Masarwah and A.G. Ahmad, On (complete) normality of m-pF subalgebras in BCK/BCI-algebras, AIMS Mathematics, 4(3)(2019), 740-750.

[9] A. Al-Masarwah and A.G. Ahmad, On some properties of doubt bipolar fuzzy H-ideals in BCK/BCI-algebras, European Journal of Pure and Applied Mathematics, 11(3)(2018), 652-670.
[10] A. Al-Masarwah, A.G. Ahmad and G. Muhiuddin, Doubt ideal theory in BCK-algebras based on $\mathcal{N}$-structures, Annals of Communications in Mathematics, 3(1)(2020), 54-62.
[11] H.M. Khalid and B. Ahmad, Fuzzy H-ideals in BCI-algebras, Fuzzy Sets and Systems, 101(1)(1999), 153-158.
[12] Y. Imai and K. Iséki, On axiom systems of theoremal calculi, Proceeding of the Japan Academy, 42(1966), 19-22.
[13] Y.B. Jun, Doubt fuzzy BCK/BCI-algebras, Soochow Journal of Mathematics, 20(3)(1994), 351-358.
[14] Y.B. Jun, K.J. Lee and S.Z. Song, $\mathcal{N}$-ideals of BCK/BCI-algerbas, Journal of the Chungcheong Mathematical Society, 22(3)(2009), 417-437.
[15] Y.B. Jun, G. Muhiuddin and A.M. Al-roqi, Ideal theory of BCK/BCI-algebras based on double-framed soft sets, Applied Mathematics and Information Sciences, 7(2013), 1879-1887.
[16] Y.B. Jun, G. Muhiuddin, M.A. Ozturk and E.H. Roh, Cubic soft ideals in BCK/BCI-algebras, Journal of Computational Analysis and Applications, 22(5)(2017), 929-940.
[17] Y.B. Jun, F. Smarandache and H. Bordbar, Neutrosophic N-structures applied to BCK/BCI-algebras, Information, (8)(2017), 128.
[18] M. Khan, S. Anis, F. Smarandache and Y.B. Jun, Neutrosophic N- structures and their applications in semigroups, Annals of Fuzzy Mathematics and Informatics, 14(2017), 583598.
[19] O.G. Xi, Fuzzy BCK-algebras, Mathematica Japonica, 36(5)(1991), 935-942.
[20] L.A. Zadeh, Fuzzy sets, Information and Control, (8)(1965), 338-353.
[21] J. Zhan and Z. Tan, Characterization of doubt fuzzy H-ideals in BCK-algebras, Soochow Journal of Mathematics, 29(3)(2003), 293-298.

ANAS AL-MASARWAH
SCHOOL OF MATHEMATICAL SCIENCES, FACULTY OF SCIENCE AND TECHNOLOGY, UNIVERSITI KEBANGSAAN MALAYSIA, 43600 UKM BANGI, SELANGOR DE, MALAYSIA
Email address: almasarwah85@gmail.com

ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 3, Number 4 (2020), 317-325
ISSN: 2582-0818

SOME MORE RESULTS ON $\epsilon - LP-$SASAKIAN MANIFOLDS ADMITTING η-RICCI SOLITONS

ABDUL HASEEB*, LOVEJOY S. DAS AND MOBIN AHMAD

ABSTRACT. The object of the present paper is to characterize $\epsilon - LP-$Sasakian manifolds with a quarter-symmetric metric connection admitting $\eta-$Ricci solitons. Finally, the existence of $\eta-$Ricci soliton in an $\epsilon - LP-$Sasakian manifold has been proved by a concrete example.

1. INTRODUCTION

The study of manifolds with indefinite metrics is of high interest in physics and relativity theory. In 1993, the concept of ϵ-Sasakian manifolds was introduced by Bejancu and Duggal [2]. Later, it was shown by Xufeng and Xiaoli [18] that these manifolds are real hypersurfaces of indefinite Kahlerian manifolds. In 2012, Prasad and Srivastava [14] have studied ϵ-Lorentzian para-Sasakian manifolds and shown its existence by an example. On the other hand, the concept of ϵ-Kenmotsu manifold was introduced by U. C. De and A. Sarkar [6] who showed that the existence of new structure on an indefinite metrics influences the curvatures. Recently, the manifolds with indefinite metrics have been studied by various authors in several ways to a different extent such as ([10], [12], [17]) and many others.

As a generalization of Ricci solitons, the notion of η-Ricci solitons was introduced by Cho and Kimura [3]. They have studied Ricci solitons of real hypersurfaces in a non-flat complex space form and defined η-Ricci solitons, which satisfies the equation

$$\pounds_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.1}$$

where S is the Ricci tensor associated to g, η is a 1-form and λ, μ are real numbers. In particular, if $\mu = 0$, then the notion of η-Ricci soliton (g, V, λ, μ) reduces to the notion of Ricci soliton (g, V, λ). Recenty, $\eta-$Ricci solitons have been studied by various authors such as ([4], [9], [11], [13], [15], [19]) and many others.

2010 *Mathematics Subject Classification.* 53C15, 53C25.
Key words and phrases. $\eta-$Ricci solitons, $\epsilon-LP-$Sasakian manifold, quarter-symmetric metric connection.
Received: December 01, 2020. Accepted: December 24, 2020. Published: December 31, 2020.
*Corresponding author.

2. Preliminaries

A differentiable manifold M of dimension n is called an ϵ-Lorentzian para-Sasakian (briefly, $\epsilon - LP-$Sasakian), if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian like metric g which satisfy

$$\phi^2 X = X + \eta(X)\xi,\ \eta(\xi) = -1, \tag{2.1}$$

$$g(\xi,\xi) = -\epsilon,\ \eta(X) = \epsilon g(X,\xi),\ \phi\xi = 0,\ \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \epsilon\eta(X)\eta(Y), \tag{2.3}$$

$$(\nabla_X \phi)Y = g(X,Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi, \tag{2.4}$$

$$\nabla_X \xi = \epsilon\phi X \tag{2.5}$$

for all X, $Y \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields on the manifold M, ϵ is -1 or 1 according to the vector field ξ being spacelike or timelike and ∇ denotes the Levi-Civita connection with respect to g.
If we put

$$\Phi(X,Y) = g(\phi X, Y) \tag{2.6}$$

for all vector fields X and Y on M, then $\Phi(X,Y)$ is a symmetric $(0,2)$ tensor field. Also since the 1-form η is closed in an $\epsilon - LP-$Sasakian manifold, so we have [14]

$$(\nabla_X \eta)(Y) = \Phi(X,Y),\ \Phi(X,\xi) = 0 \tag{2.7}$$

for all $X, Y \in \chi(M)$.
Moreover, the curvature tensor R, the Ricci tensor S and the Ricci operator Q in an $\epsilon - LP-$Sasakian manifold with the Levi-Civita connection satisfy the following equations [14]:

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.8}$$

$$R(\xi,X)Y = \epsilon g(X,Y)\xi - \eta(Y)X, \tag{2.9}$$

$$R(\xi,X)\xi = -R(X,\xi)\xi = X + \eta(X)\xi, \tag{2.10}$$

$$\eta(R(X,Y)Z) = \epsilon[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.11}$$

$$S(X,\xi) = (n-1)\eta(X), Q\xi = \epsilon(n-1)\xi, \tag{2.12}$$

where $X, Y, Z \in \chi(M)$ and $g(QX,Y) = S(X,Y)$.
We note that if $\epsilon = 1$ and the structure vector field ξ is timelike, then an $\epsilon - LP-$Sasakian manifold is usual $LP-$Sasakian manifold.

Definition 2.1. An $\epsilon - LP-$Sasakian manifold is said to be an η-Einstein manifold if its non-vanishing Ricci tensor S of type $(0,2)$ satisfies [20]

$$S(Y,Z) = \alpha g(Y,Z) + \beta\eta(Y)\eta(Z) \tag{2.13}$$

where α and β are the smooth functions on M. If $\beta = 0$, then M is said to be an Einstein manifold.

3. Curvature tensor in an $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection

A linear connection $\bar{\nabla}$ in a Riemannian manifold M is said to be a quarter-symmetric connection [7] if the torsion tensor T of the connection $\bar{\nabla}$ is of the form

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] = \eta(Y)\phi X - \eta(X)\phi Y, \tag{3.1}$$

where η is a 1-form and ϕ is a $(1,1)$ tensor field. If moreover, a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y,Z) = 0 \tag{3.2}$$

for all $X, Y, Z \in \chi(M)$, then the connection $\bar{\nabla}$ is said to be a quarter-symmetric metric, otherwise it is said to be a quarter-symmetric non-metric.

A quarter-symmetric metric connection $\bar{\nabla}$ in an $\epsilon - LP-$Sasakian manifold is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - \epsilon g(\phi X, Y)\xi. \tag{3.3}$$

A quarter symmetric metric connection have been studied by various authors such as Ahmad et al. [1], De and Mondal [5], Singh and Pandey [16] and many others.

If $\bar{R}$ and $\bar{S}$, respectively, are the curvature tensor and the Ricci tensor of a quarter-symmetric metric connection $\bar{\nabla}$ in an $\epsilon - LP-$Sasakian manifold M, then we have [9]

$$\bar{R}(X,Y)Z = R(X,Y)Z + (2-\epsilon)[g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X] \tag{3.4}$$
$$+\epsilon\eta(Z)[\eta(Y)X - \eta(X)Y] + [\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]\xi,$$
$$\bar{R}(X,Y)\xi = (1-\epsilon)[\eta(Y)X - \eta(X)Y], \tag{3.5}$$
$$\bar{R}(\xi,X)Y = -(1-\epsilon)[g(X,Y)\xi + \eta(Y)X], \tag{3.6}$$
$$\bar{R}(\xi,X)\xi = (1-\epsilon)[X + \eta(X)\xi], \tag{3.7}$$
$$\bar{S}(Y,Z) = S(Y,Z) + (1-\epsilon)g(Y,Z) + (n\epsilon - 1)\eta(Y)\eta(Z) - (2-\epsilon)g(\phi Y, Z)\psi, \tag{3.8}$$
$$\bar{S}(X,\xi) = (1-\epsilon)(n-1)\eta(X),\ \bar{Q}\xi = -(1-\epsilon)(n-1)\xi \tag{3.9}$$

for all $X, Y, Z \in \chi(M)$.

Lemma 3.1. *Let M be an n-dimensional $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection. Then we have*

$$(\bar{\nabla}_X \phi)Y = (1-\epsilon)(g(X,Y)\xi - \eta(Y)X - 2\eta(X)\eta(Y)\xi), \tag{3.10}$$
$$\bar{\nabla}_X \xi = -(1-\epsilon)\phi X, \tag{3.11}$$
$$(\bar{\nabla}_X \eta)Y = (1-\epsilon)g(\phi X, Y), \tag{3.12}$$
$$(\bar{\mathcal{L}}_\xi g)(X,Y) = 2\epsilon g(\phi X, Y) \tag{3.13}$$

for all $X, Y \in \chi(M)$.

Proof. By the covariant differentiation of ϕY with respect to X, we have

$$\bar{\nabla}_X \phi Y = (\bar{\nabla}_X \phi)Y + \phi(\bar{\nabla}_X Y)$$

which in view of (3.3) takes the form

$$(\bar{\nabla}_X \phi)Y = (\nabla_X \phi)Y - \epsilon g(\phi X, \phi Y)\xi - \eta(Y)\phi^2 X. \tag{3.14}$$

By making use of (2.1), (2.3) and (2.4) in the last equation, (3.10) follows.
Next, we replace $Y = \xi$ in (3.3) and using (2.2) we find

$$\bar{\nabla}_X \xi = \nabla_X \xi + \eta(\xi)\phi X. \tag{3.15}$$

By using (2.1) and (2.5) in (3.15), we get (3.11).
In order to prove (3.12), we differentiate $\eta(Y)$ covariantly with respect to X and using (3.2), we find

$$(\bar{\nabla}_X \eta)(Y) = \epsilon g(Y, \bar{\nabla}_X \xi) \tag{3.16}$$

which in view of (3.11) gives (3.12).
In view of (3.1), the expression $(\bar{\mathcal{L}}_\xi g)(X,Y) = \bar{\mathcal{L}}_\xi g(X,Y) - g(\bar{\mathcal{L}}_\xi X, Y) - g(X, \bar{\mathcal{L}}_\xi Y)$ takes the form

$$(\bar{\mathcal{L}}_\xi g)(X,Y) = \bar{\nabla}_\xi g(X,Y) - g[\bar{\nabla}_\xi X - \bar{\nabla}_X \xi - \phi X, Y] + g[X, \bar{\nabla}_\xi Y - \bar{\nabla}_Y \xi - \phi Y]$$

in which using (3.11), we obtain (3.13). □

In [9], A. Haseeb and R. Prasad have studied $\eta-$Ricci solitons in an $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection and proved the following:

Lemma 3.2. *If (g, ξ, λ, μ) is an $\eta-$Ricci soliton in an $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection, then*

$$\bar{S}(Y, Z) = -\epsilon g(\phi Y, Z) - \lambda g(Y, Z) - \mu\eta(Y)\eta(Z), \tag{3.17}$$

$$\lambda - \epsilon\mu = (1 - \epsilon)(n - 1) \tag{3.18}$$

for all $Y, Z \in \chi(M)$.

4. $\eta-$Ricci solitons in $\epsilon - LP-$Sasakian manifolds with a quarter-symmetric metric connection admitting Codazzi type of Ricci tensor

In this section we consider $\eta-$Ricci solitons in $\epsilon - LP-$Sasakian manifolds with a quarter-symmetric metric connection admitting Codazzi type of Ricci tensor. A. Gray [8] introduced the notion of cyclic parallel Ricci tensor and Codazzi type of Ricci tensor.

Definition 4.1. An $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection is said to have Codazzi type of Ricci tensor if its Ricci tensor $\bar{S}$ of type $(0, 2)$ is non-zero and satisfies the following condition

$$(\bar{\nabla}_X\bar{S})(Y, Z) = (\bar{\nabla}_Y\bar{S})(Z, X) \tag{4.1}$$

for all $X, Y, Z \in \chi(M)$.

Taking covariant derivative of (3.17) and making use of (2.2), (3.10) and (3.12), we find

$$\begin{aligned}(\bar{\nabla}_X\bar{S})(Y, Z) = (1 - \epsilon)[\epsilon g(X, Y)\eta(Z) - \eta(Y)g(X, Z) - 2\epsilon\eta(X)\eta(Y)\eta(Z)]\\ -(1 - \epsilon)\mu[g(\phi X, Y)\eta(Z) + g(\phi X, Z)\eta(Y)].\end{aligned} \tag{4.2}$$

If the Ricci tensor $\bar{S}$ is of Codazzi type, then

$$(\bar{\nabla}_X\bar{S})(Y, Z) = (\bar{\nabla}_Y\bar{S})(Z, X). \tag{4.3}$$

Using (4.2) in (4.3), we have

$$g(X, Y)\eta(Z) - g(Y, Z)\eta(X) + \mu[g(\phi X, Y)\eta(Z) - g(\phi Y, Z)\eta(X)] = 0, \ (1 - \epsilon) \neq 0,$$

which by taking $Z = \xi$ and using (2.2) reduces to

$$g(X, Y) + \epsilon\eta(X)\eta(Y) + \mu g(\phi X, Y) = 0. \tag{4.4}$$

By putting $Y = \phi Y$ in (4.4), we find

$$g(X, \phi Y) + \mu g(\phi X, \phi Y) = 0. \tag{4.5}$$

Replacing X by ϕX in (4.5), we have

$$g(\phi X, \phi Y) + \mu g(X, \phi Y) = 0. \tag{4.6}$$

Now adding (4.5) and (4.6), we obtain

$$(1 + \mu)[g(X, \phi Y) + g(\phi X, \phi Y)] = 0 \tag{4.7}$$

from which it follows that $\mu = -1$. From the relation (3.18), we get $\lambda = n(1 - \epsilon) - 1$. Thus we have the following:

Theorem 4.1. *Let (g, ξ, λ, μ) be an $\eta-$Ricci soliton in an n-dimensional $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection and if the manifold has Ricci tensor of Codazzi type, then $\mu = -1$ and $\lambda = n(1 - \epsilon) - 1$.*

5. $\eta-$RICCI SOLITONS IN $\epsilon - LP-$SASAKIAN MANIFOLDS WITH A QUARTER-SYMMETRIC METRIC CONNECTION ADMITTING CYCLIC PARALLEL RICCI TENSOR

Definition 5.1. An $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection is said to have cyclic parallel Ricci tensor if its Ricci tensor $\bar{S}$ of type $(0, 2)$ is non-zero and satisfies the following condition [8]

$$(\bar{\nabla}_X \bar{S})(Y, Z) + (\bar{\nabla}_Y \bar{S})(Z, X) + (\bar{\nabla}_Z \bar{S})(X, Y) = 0 \tag{5.1}$$

for all $X, Y, Z \in \chi(M)$.

Let (g, ξ, λ, μ) be an $\eta-$Ricci soliton in an $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection and the manifold M has cyclic parallel Ricci tensor, then (5.1) holds. Taking covariant derivative of (3.17) and making use of (2.2), (3.10) and (3.12), we find

$$\begin{aligned}(\bar{\nabla}_X \bar{S})(Y, Z) = (1-\epsilon)[\epsilon g(X, Y)\eta(Z) - \eta(Y)g(X, Z) - 2\epsilon\eta(X)\eta(Y)\eta(Z)] \\ -(1-\epsilon)\mu[g(\phi X, Y)\eta(Z) + g(\phi X, Z)\eta(Y)].\end{aligned} \tag{5.2}$$

Similarly, we have

$$\begin{aligned}(\bar{\nabla}_Y \bar{S})(Z, X) = (1-\epsilon)[\epsilon g(Y, Z)\eta(X) - \eta(Z)g(Y, X) - 2\epsilon\eta(X)\eta(Y)\eta(Z)] \\ -(1-\epsilon)\mu[g(\phi Y, Z)\eta(X) + g(\phi Y, X)\eta(Z)],\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}(\bar{\nabla}_Z \bar{S})(X, Y) = (1-\epsilon)[\epsilon g(Z, X)\eta(Y) - \eta(X)g(Z, Y) - 2\epsilon\eta(X)\eta(Y)\eta(Z)] \\ -(1-\epsilon)\mu[g(\phi Z, X)\eta(Y) + g(\phi Z, Y)\eta(X)].\end{aligned} \tag{5.4}$$

By using (5.2)-(5.4) in (5.1), we obtain

$$\begin{aligned}-2[g(X, Y)\eta(Z) + g(Y, Z)\eta(X) + g(Z, X)\eta(Y)] + 6\eta(X)\eta(Y)\eta(Z) \\ -2\mu[g(\phi X, Y)\eta(Z) + g(\phi Y, Z)\eta(X) + g(\phi Z, X)\eta(Y)] = 0, \quad (1-\epsilon) \neq 0\end{aligned}$$

which by taking $Z = \xi$ reduces to

$$g(\phi X, \phi Y) + \mu g(\phi X, Y) = 0. \tag{5.5}$$

Replacing Y by ϕY in (5.5), we have

$$g(\phi X, Y) + \mu g(\phi X, \phi Y) = 0. \tag{5.6}$$

By adding (5.5) and (5.6), we obtain

$$(1 + \mu)[g(\phi X, Y) + g(\phi X, \phi Y)] = 0 \tag{5.7}$$

from which it follows that $\mu = -1$. From the relation (3.18), we get $\lambda = n(1 - \epsilon) - 1$. Thus we have the following:

Theorem 5.1. *Let (g, ξ, λ, μ) be an $\eta-$Ricci soliton in an n-dimensional $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection and if the manifold has cyclic parallel Ricci tensor, then $\mu = -1$ and $\lambda = n(1 - \epsilon) - 1$.*

Definition 5.2. An $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection is said to have cyclic $\eta-$recurrent Ricci tensor, if

$$\begin{aligned}(\bar{\nabla}_X \bar{S})(Y, Z) + (\bar{\nabla}_Y \bar{S})(Z, X) + (\bar{\nabla}_Z \bar{S})(X, Y) \\ = \eta(X)\bar{S}(Y, Z) + \eta(Y)\bar{S}(Z, X) + \eta(Z)\bar{S}(X, Y)\end{aligned} \tag{5.8}$$

for all $X, Y, Z \in \chi(M)$.

Suppose that the manifold M with a quarter-symmetric metric connection has cyclic $\eta-$recurrent Ricci tensor, then (5.8) holds. By using (3.17) and (5.2)-(5.4) in (5.8), we get

$$(\lambda-2(1-\epsilon))(g(X,Y)\eta(Z)+g(Y,Z)\eta(X)+g(Z,X)\eta(Y))+(6(1-\epsilon)+3\mu)\eta(X)\eta(Y)\eta(Z)$$
$$+(\epsilon-2(1-\epsilon)\mu)(g(\phi X,Y)\eta(Z)+g(\phi Y,Z)\eta(X)+g(\phi Z,X)\eta(Y))=0$$

which by putting $Y=Z=\xi$ gives $\mu=\epsilon\lambda$. Thus we have the following:

Theorem 5.2. *Let (g,ξ,λ,μ) be an $\eta-$Ricci soliton in an n-dimensional $\epsilon-LP-$Sasakian with a quarter-symmetric metric connection and if the manifold has cyclic $\eta-$recurrent, then $\mu=\epsilon\lambda$.*

6. ϕ-Ricci symmetric $\eta-$Ricci solitons in $\epsilon-LP-$Sasakian manifolds with a quarter-symmetric metric connection

Definition 6.1. An $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection is said to be ϕ-Ricci symmetric if the Ricci operator $\bar{Q}$ satisfies

$$\phi^2((\bar{\nabla}_X\bar{Q})(Y))=0 \tag{6.1}$$

for any $X,Y\in\chi(M)$.

Let (g,ξ,λ,μ) be an $\eta-$Ricci soliton in an n-dimensional $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection and the manifold M is ϕ-Ricci symmetric. Then (6.1) holds, which in view of (2.1) yields

$$(\bar{\nabla}_X\bar{Q})Y+\eta((\bar{\nabla}_X\bar{Q})Y)\xi=0. \tag{6.2}$$

Taking the inner product of (6.2) with Z and using (2.2), we find

$$g((\bar{\nabla}_X\bar{Q})Y,Z)+\epsilon\eta((\bar{\nabla}_X\bar{Q})Y)\eta(Z)=0$$

which can be written as

$$g(\bar{\nabla}_X\bar{Q}Y,Z)-\bar{S}(\bar{\nabla}_XY,Z)+\epsilon\eta((\bar{\nabla}_X\bar{Q})Y)\eta(Z)=0. \tag{6.3}$$

Now putting $Y=\xi$ in (6.3) and using (3.9) and (3.11), we get

$$(1-\epsilon)^2(n-1)g(\phi X,Z)+(1-\epsilon)\bar{S}(\phi X,Z)+\epsilon\eta((\bar{\nabla}_X\bar{Q})\xi)\eta(Z)=0.$$

Replacing Z by ϕZ in the last equation and using (2.2), we get

$$(1-\epsilon)(n-1)g(\phi X,\phi Z)+\bar{S}(\phi X,\phi Z)=0,\quad (1-\epsilon)\neq 0. \tag{6.4}$$

By virtue of (3.17), (6.4) takes the form

$$[\lambda-(1-\epsilon)(n-1)]g(\phi X,\phi Z)+\epsilon(X,\phi Z)=0. \tag{6.5}$$

Replacing X by ϕX in (6.5), we have

$$[\lambda-(1-\epsilon)(n-1)]g(X,\phi Z)+\epsilon(\phi X,\phi Z)=0. \tag{6.6}$$

By adding (6.5) and (6.6), we get

$$[\lambda-(1-\epsilon)(n-1)+\epsilon](g(X,\phi Z)+\epsilon(\phi X,\phi Z))=0 \tag{6.7}$$

from which it follows that $\lambda=(1-\epsilon)(n-1)-\epsilon$. From the relation (3.18), we get $\mu=-1$. Thus we have the following theorem:

Theorem 6.1. *Let (g,ξ,λ,μ) be an $\eta-$Ricci soliton in an n-dimensional $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection and if the manifold is ϕ-Ricci symmetric, then $\lambda=(1-\epsilon)(n-1)-\epsilon$ and $\mu=-1$.*

Now from the relations (2.3), (3.18) and (6.5), we obtain

$$\bar{S}(X,Z) = -(n-1)(1-\epsilon)g(X,Z). \tag{6.8}$$

Thus we have

Corollary 6.2. *An n-dimensional ϕ-Ricci symmetric $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection admitting an η-Ricci soliton (g, ξ, λ, μ) is an Einstein manifold of the form* (6.8).

7. The $\eta-$parallel $\phi-$tensor in $\epsilon - LP-$Sasakian manifolds with a quarter-symmetric metric connection

In this section we study the $\eta-$parallel $\phi-$tensor in $\epsilon - LP-$Sasakian manifolds with a quarter-symmetric metric connection. If the $(1,1)-$tensor ϕ is $\eta-$parallel, then we have

$$g((\bar{\nabla}_X \phi)Y, Z) = 0 \tag{7.1}$$

for any $X, Y, Z \in \chi(M)$.
By using (3.10) in (7.1), we have

$$(1-\epsilon)g(g(X,Y)\xi - \eta(Y)X - 2\eta(X)\eta(Y)\xi, Z) = 0.$$

By using (2.2) in the last equation, we find

$$(1-\epsilon)(\epsilon g(X,Y)\eta(Z) - \eta(Y)g(X,Z) - 2\epsilon\eta(X)\eta(Y)\eta(Z)) = 0. \tag{7.2}$$

Putting $Z = \xi$ in (7.2), we get

$$(1-\epsilon)(g(X,Y) - \eta(X)\eta(Y)) = 0 \tag{7.3}$$

from which it follows that either
(i) $\epsilon = 1$(i.e., the vector field ξ is timelike), or (ii) $g(X,Y) = \eta(X)\eta(Y)$.
Putting $X = \bar{Q}X$ in (ii), we obtain

$$\bar{S}(X,Y) = -(1-\epsilon)(n-1)\eta(X)\eta(Y). \tag{7.4}$$

In view of (3.17), (7.4) takes the form

$$\epsilon g(\phi Y, Z) + \lambda g(Y,Z) + [\mu - (1-\epsilon)(n-1)]\eta(Y)\eta(Z) = 0. \tag{7.5}$$

Taking $Y = Z = \xi$ in (7.5) and using (2.1) and (2.2), we obtain

$$\lambda - \epsilon\mu = (1-\epsilon)(n-1). \tag{7.6}$$

Thus we have the following theorem:

Theorem 7.1. *If in an n-dimensional $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection the tensor ϕ is $\eta-$parallel, then either the vector field ξ is timelike or the manifold is a special type of an $\eta-$Einstein manifold of the form* (7.4) *and the scalars λ and μ are related by $\lambda - \epsilon\mu = (1-\epsilon)(n-1)$.*

If μ=0, then (7.6) reduces to $\lambda = (1-\epsilon)(n-1)$. In this case we have

Corollary 7.2. *If (g, ξ, λ) is a Ricci soliton in an n-dimensional $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection with $\eta-$parallel $\phi-$tensor, then the Ricci soliton on M is expanding according to the vector field ξ being spacelike.*

Example. We consider the 3-dimensional manifold $M = \{(x,y,z) \in R^3\}$, where (x,y,z) are the standard coordinates in R^3. Let e_1, e_2 and e_3 be the vector fields on M given by

$$e_1 = coshx_3\frac{\partial}{\partial x_1} + sinhx_3\frac{\partial}{\partial x_2},\ e_2 = sinhx_3\frac{\partial}{\partial x_1} + coshx_3\frac{\partial}{\partial x_2},\ e_3 = \frac{\partial}{\partial x_3} = \xi.$$

Let g be the Lorentzian like (semi-Riemannian) metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = 1,\ g(e_3, e_3) = -\epsilon,\ g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0.$$

Let η be the 1-form on M defined by $\eta(X) = \epsilon g(X, e_3) = \epsilon g(X, \xi)$ for all $X \in \chi(M)$. Let ϕ be the $(1, 1)$ tensor field on M defined by

$$\phi e_1 = -\epsilon e_2,\ \phi e_2 = -\epsilon e_1,\ \phi e_3 = 0.$$

Using Koszul's formula for the metric g, we can easily calculate

$$\nabla_{e_1} e_1 = 0,\ \nabla_{e_2} e_1 = -\epsilon e_3,\ \nabla_{e_3} e_1 = 0,\ \nabla_{e_1} e_2 = -\epsilon e_3,\ \nabla_{e_2} e_2 = 0, \tag{7.7}$$
$$\nabla_{e_3} e_2 = 0,\ \nabla_{e_1} e_3 = -e_2,\ \nabla_{e_2} e_3 = -e_1,\ \nabla_{e_3} e_3 = 0.$$

Also, one can easily verify that

$$\nabla_X \xi = \epsilon\phi X \quad \text{and} \quad (\nabla_X \phi)Y = g(X, Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi.$$

Thus the manifold M is an $\epsilon - LP-$Sasakian manifold. From (3.3) and (7.7), we obtain

$$\bar{\nabla}_{e_1} e_1 = 0,\ \bar{\nabla}_{e_2} e_1 = (1-\epsilon)e_3,\ \bar{\nabla}_{e_3} e_1 = 0,\ \bar{\nabla}_{e_1} e_2 = (1-\epsilon)e_3 \tag{7.8}$$
$$\bar{\nabla}_{e_2} e_2 = 0,\ \bar{\nabla}_{e_3} e_2 = 0,\ \bar{\nabla}_{e_1} e_3 = 0,\ \bar{\nabla}_{e_2} e_3 = 0,\ \bar{\nabla}_{e_3} e_3 = 0.$$

It is known that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{7.9}$$

By using the above results, we can easily obtain the components of the curvature tensors as follows:

$$R(e_1, e_2)e_1 = \epsilon e_2,\ R(e_1, e_2)e_2 = -\epsilon e_1,\ R(e_1, e_2)e_3 = 0, \tag{7.10}$$
$$R(e_2, e_3)e_1 = 0,\ R(e_2, e_3)e_2 = -\epsilon e_3,\ R(e_2, e_3)e_3 = -e_2,$$
$$R(e_1, e_3)e_1 = -\epsilon e_3,\ R(e_1, e_3)e_2 = 0,\ R(e_1, e_3)e_3 = -e_1,$$

and

$$\bar{R}(e_1, e_2)e_1 = \epsilon e_2,\ \bar{R}(e_1, e_2)e_2 = 2(1-\epsilon)e_1,\ \bar{R}(e_1, e_2)e_3 = 0, \tag{7.11}$$
$$\bar{R}(e_2, e_3)e_1 = 0,\ \bar{R}(e_2, e_3)e_2 = (1-\epsilon)e_3,\ \bar{R}(e_2, e_3)e_3 = -(1-\epsilon)e_2,$$
$$\bar{R}(e_1, e_3)e_1 = (1-\epsilon)e_3,\ \bar{R}(e_1, e_3)e_2 = 0,\ \bar{R}(e_1, e_3)e_3 = -(1-\epsilon)e_1.$$

From these curvature tensors, we can easily calculate

$$S(e_1, e_1) = S(e_2, e_2) = 0,\ S(e_3, e_3) = -2. \tag{7.12}$$

$$\bar{S}(e_1, e_1) = \bar{S}(e_2, e_2) = (1-\epsilon),\ \bar{S}(e_3, e_3) = -2(1-\epsilon). \tag{7.13}$$

Now from (3.17) and (7.13), we obtain $\lambda = -(1-\epsilon)$ and $\mu = 3(1-\epsilon)$. Therefore the data (g, ξ, λ, μ) for $\lambda = -(1-\epsilon)$ and $\mu = 3(1-\epsilon)$ defines an $\eta-$Ricci soliton on M.

REFERENCES

[1] Ahmad, M., Jun, J. B. and Haseeb, A., Hypersurfaces of almost r-paracontact Riemannian manifold with a quarter-symmetric metric connection, Bull. Korean Math. Soc., 46(2009), no. 3, 477-487.

[2] Bejancu, A. and Duggal, K. L., Real hypersurfaces of indefinite Kaehler manifolds, Int. J. Math. Math. Sci., 16(1993), 545-556.

[3] Cho, J. T. and Kimura, M., Ricci solitons and real hypersurfaces in a complex space form, Tohoku Math. J., 61(2009), no. 2, 205-212.

[4] De, K. and De, U. C., η-Ricci solitons on Kenmotsu 3-manifolds, An. Univ. Vest Timi. Ser. Mat.-Inform., 56 (2018), 51-63.

[5] De, U. C. and Mondal, A. K., Quarter-symmetric metric connection on 3-dimensional quasi-Sasakian manifolds, SUT J. Math., 46(2010), 35-52.

[6] De, U. C. and Sarkar, A., On $\epsilon-$Kenmotsu manifold, Hardonic J. 32(2009), no.2, 231-242.

[7] Golab, S., On semi-symmetric and quarter-symmetric linear connections, Tensor (N.S.), 29(1975), 249-254.

[8] Gray, A., Einstein-like manifolds which are not Einstein, Geom. Dedicata, 7(1978), 259-280.
[9] Haseeb, A. and Prasad, R., η-Ricci solitons on $\epsilon - LP-$Sasakian manifolds with a quarter-symmetric metric connection, Honam Math. J., 41 (2019), 539-558.
[10] Haseeb, A. and Prasad, R., $*-$Conformal $\eta-$Ricci solitons in $\epsilon-LP-$Sasakian manifolds, Publ. De L'Inst. Math., Nouvelle serie, tome 108(122) (2020), 89-100.
[11] Haseeb, A. and De, U. C., η-Ricci solitons in $\epsilon-$Kenmotsu manifolds, J. Geom., 110 (2019), 12 pages.
[12] Kumar, R., Rani, R. and Nagaich, R. K., On sectional curvature of $\epsilon-$Sasakian manifolds, IJMMS, (2007), Article ID: 93562, 10 pages.
[13] Prakasha, D. G. and Hadimani, B. S., η-Ricci solitons on para-Sasakian manifolds, J. Geom., 108 (2017), 383-392.
[14] Prasad, R. and Srivastava, V., On ϵ-Lorentzian para-Sasakian manifolds, Commun. Korean Math. Soc., 27(2012), 297-306.
[15] Singh, A. and Kishor, S., Some types of η-Ricci solitons on Lorentzian para-Sasakian manifolds, Facta Universitatis (NIS), 33(2018), no. 2, 217-230.
[16] Singh, R. N. and Pandey, S. K., On quarter-symmetric metric connection in an LP-Sasakian manifold, Thai J. Math., 12(2014), 357-371.
[17] Venkatesha, S . V. Vishnuvardhana, $\epsilon-$Kenmotsu manifolds admitting a semi-symmetric metric connection, Ital. J. Pure Appl. Math., 38 (2017), 615-623.
[18] Xufeng, X. and Xiaoli, C., Two theorems on ϵ-Sasakian manifolds, Int. J. Math. Math. Sci., 21(1998), 249-54.
[19] Yadav, S. K., Chaubey, S. K. and Suthar, D. L., Some results on η-Ricci solitons on $(LCS)_n$- manifolds, Surveys in Mathematics and its applications., 13 (2018), 237-250.
[20] Yano, K. and Kon, M., Structures on Manifolds, Series in Pure Math., Vol. 3, World Sci., 1984.

ABDUL HASEEB
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, JAZAN UNIVERSITY, JAZAN, KINGDOM OF SAUDI ARABIA.
Email address: malikhaseeb80@gmail.com, haseeb@jazanu.edu.sa

LOVEJOY S. DAS
DEPARTMENT OF MATHEMATICS, KENT STATE UNIVERSITY, KENT, USA.
Email address: ldas@kent.edu

MOBIN AHMAD
DEPARTMENT OF MATHEMATICS, INTEGRAL UNIVERSITY, LUCKNOW, INDIA.
Email address: mobinahmad@rediffmail.com

ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 3, Number 4 (2020), 326-332
ISSN: 2582-0818

ROUGH APPROXIMATIONS OF INTERVAL ROUGH FUZZY IDEALS IN GAMMA-SEMIGROUPS

V. S. SUBHA* AND P. DHANALAKSHMI

ABSTRACT. In this paper, we introduce the notions of interval rough fuzzy Γ-ideals, bi-Γ-ideal, prime-Γ- ideal and prime-bi-Γ-ideal in Γ-semigroup and establish some interesting properties of these structures.

1. INTRODUCTION

In 1965, the concept of fuzzy sets was introduced by Zadeh[10] and paved the way to study uncertainty problems. Applications of fuzzy set theory have been found in various fields. Sen [7] definded the Γ-semigroup in 1986. The notion of rough set was introduced by Pawlak[4] as a new tool for reasoning about data. Rough set theory is a powerful tool to deal with imperfect data. The rough set theory is an extension of set theory. The main idea of rough set corresponds to the concepts of lower and upper approximations of a set. The lower approximation of a given set is the union of all equivalence classes which are subsets of the set, and the upper approximation is the union of all equivalence classes which have a nonempty intersection with the set. Zadeh[9] introduced the notion of interval-valued fuzzy sets as a generalization of fuzzy sets in 1975, ie., a fuzzy subset with an interval-valued membership function. Later by combining rough sets and fuzzy sets, the notion of rough fuzzy sets were introduced by Dubois and Prade [2]in 1990. The notion of interval valued rough fuzzy sets in semigroups was introduced by Subha et al.[5].

Throughout this paper let us denote S as Γ-semigroup, ∂ as complete congruence relation and Ω as interval valued fuzzy set.

2. PRELIMINARIES

In this section we list some concepts that are required in the development of our work.

Definition 2.1. Let S be a Γ semigroup and ∂ congruence relation on S. The pair (S, ∂) is called an approximation space. Let Ω be any nonempty subset of S. The sets

$$\partial^l(\Omega) = \{x \in S/[x]_\partial \subseteq \Omega\} \text{ and } \partial^u(\Omega) = \{x \in S/[x]_\partial \cap \Omega \neq \phi\}$$

2010 *Mathematics Subject Classification.* 08A72, 20M12, 03E72.
Key words and phrases. Rough sets; Interval fuzzy sets; Interval rough fuzzy set.
Received: December 09, 2020. Accepted: December 26, 2020. Published: December 31, 2020.
*Corresponding author.

are called the lower and upper approximations of Ω. Then $\partial(\Omega) = (\partial^l(\Omega), \partial^u(\Omega))$ is called rough set in (S, ∂).

Definition 2.2. Let Ω be a fuzzy subset of S. The fuzzy subsets of S defined by

$$\partial^u(\Omega)(x) = \bigvee_{a\in[x]_\partial} \Omega(a) \text{ and } \partial^l(\Omega)(x) = \bigwedge_{a\in[x]_\partial} \Omega(a)$$

are called respectively, the upper and lower approximations of the fuzzy set Ω.
$\partial(\Omega) = (\partial^l(\Omega), \partial^u(\Omega))$ is called a rough fuzzy set of Ω with respect to ∂ if $\partial^l(\Omega) \neq \partial^u(\Omega)$.

Definition 2.3. Let Ω be an IF subset of S and $[\lambda_1, \lambda_2] \in D[0, 1]$, we call $(\Omega, [\lambda_1, \lambda_2]) = \{x \in X : \Omega^-(x) \geq \lambda_1, \Omega^+(x) \geq \lambda_2\}$ and
$(\Omega, (\lambda_1, \lambda_2)) = \{x \in X : \Omega^-(x) > \lambda_1, \Omega^+(x) > \lambda_2\}$ the $[\lambda_1, \lambda_2]-$ level set of Ω and $(\lambda_1, \lambda_2)-$ level set of A, respectively.

Let S be the finite and nonempty set called the universe and assume that S is a Γ-semigroup. Let Ω be an IF subset of S and let ∂ be the complete congruence relation on S. Let $\partial^l(\Omega)$ and $\partial^u(\Omega)$ be the IF subset of S defined by,

$$\partial^l(\Omega)(x) = [\wedge\Omega^-(y); y \in [x]_\partial, \wedge\Omega^+(y); y \in [x]_\partial]$$
$$\partial^u(\Omega)(x) = [\vee\Omega^-(y); y \in [x]_\partial, \vee\Omega^+(y); y \in [x]_\partial]$$

Then $\partial(\Omega) = (\partial^l(\Omega), \partial^u(\Omega))$ is called an IRF set if $\partial^l(\Omega) \neq \partial^u(\Omega)$.

Theorem 2.1. [5] *Let ∂ be a congruence relation on S. If Ω is an IF subset of S and $[\lambda_1, \lambda_2] \in D[0, 1]$, then*
(i) $(\partial^l(\Omega), [\lambda_1, \lambda_2]) = \partial^l(\Omega, [\lambda_1, \lambda_2])$
(ii) $(\partial^u(\Omega), (\lambda_1, \lambda_2)) = \partial^u(\Omega, (\lambda_1, \lambda_2))$.

Theorem 2.2. [8] *Let ∂ be a complete congruence on S and Ω prime ideal of S. Then the following statements are true.*
(i) *If $\partial^l(\Omega) \neq \phi$, then Ω is a ∂-lower rough prime ideal of S.*
(ii) *Ω is a ∂-upper rough prime ideal of S.*

3. Interval Rough fuzzy Γ-Ideals, Interval Rough fuzzy Bi-Γ-Ideal and Interval Rough fuzzy Sub-Γ-semigroup ($IRFgI$, $IRFBgI$ and $IRFSgS$)

In this section we introduce the concept of IRF set of S. We also introduce the concept of $IRFgI$, $IRFSgS$ and $IRFBgI$ of S. Example of $IRFgI$ is discussed.
An IF subset of S is called an $IRFgI$ of S if it is both upper and lower $IRFgI$ of S.

Theorem 3.1. *If Ω is an $IFgI$ of S, then $\partial^u(\Omega)$ is an $IFgI$ of S.*

Proof. Assume that Ω is an IF left gI of S, then for all $x, y \in S$, we have $\Omega(x\gamma y) \geq \Omega(y)$.

$$\begin{aligned}
\partial^u(\Omega)(x\gamma y) &= \bigvee_{p\gamma q\in[x\gamma y]_\partial} \Omega(p\gamma q) \\
&\geq \bigvee_{p\in[x]_\partial\gamma q\in[y]_\partial} \Omega(p\gamma q) \\
&\geq \bigvee_{q\in[y]_\partial} \Omega(q) \\
&= \partial^u(\Omega)(y)
\end{aligned}$$

Hence $\partial^u(\Omega)$ is an $IFgI$ of S. $\square$

Theorem 3.2. *If Ω is an $IFgI$ of S, then $\partial^u(\Omega)$ is an $IFgI$ of S.*

Proof. Similar to 3.1. □

Theorem 3.3. *If Ω is an $IFgI$ of S, then $\partial^u(\Omega)$ is $IRFgI$ of S.*

Proof. By applying Theorems 3.1 and 3.2 we get the result. □

Example 3.1. Let $S = \{e, a, b\}$ be a Γ-semigroup and $\Gamma = \{\gamma\}$ with the following multiplication table

γ	e	a	b
e	e	e	e
a	e	a	e
b	e	e	b

Let ∂ be a complete congruence relation S and the equivalence classes of S are $\{\{e\}, \{a, b\}\}$. Also an IF subset of S are defined by $\Omega(e) = [1, 1]$, $\Omega(a) = [.8, .9]$ and $\Omega(b) = [.7, .8]$. Then Ω is an $IRFgI$ of S.

Theorem 3.4. *Let Ω be an IF subset of S. Then Ω is an $IFgI$ of S if and only if for all $[\lambda_1, \lambda_2] \in D[0, 1]$ then $(\Omega, [\lambda_1, \lambda_2])(resp., (\Omega, (\lambda_1, \lambda_2)) \neq \phi, (\Omega, (\lambda_1, \lambda_2)))$ is a gI of S.*

Proof. Let us assume that Ω is an $IFgI$ of S. Assume $(\Omega, [\lambda_1, \lambda_2]) \neq \phi$.
Let $x \in (\Omega, [\lambda_1, \lambda_2])$, $y \in S$ and $\gamma \in \Gamma$. Then $\Omega(x) \geq [\lambda_1, \lambda_2]$. Since Ω is an $IFgI$ of S then $\Omega(x\gamma y) \geq \Omega(x) \wedge \Omega(y) \geq [\lambda_1, \lambda_2]$ implies $x\gamma y \in (\Omega, [\lambda_1, \lambda_2])$. Similarly $y\gamma x \in (\Omega, [\lambda_1, \lambda_2])$. Hence $(\Omega, [\lambda_1, \lambda_2])$ is an gI of S.
Conversely, let us take $[\lambda_1, \lambda_2] \in D[0, 1]$ if $(\Omega, [\lambda_1, \lambda_2]) \neq \phi$, then $(\Omega, [\lambda_1, \lambda_2])$ is an gI of S. Let $x, y \in S$ and $\gamma \in \Gamma$. Then we have two cases
Case(I): If $\Omega(x) \leq \Omega(y)$ and let $[\lambda_1, \lambda_2] = \Omega(x)$. Then $x \in (\Omega, [\lambda_1, \lambda_2])$.
By assumption $(\Omega, [\lambda_1, \lambda_2])$ is an gI of S. So $x\gamma y \in (\Omega, [\lambda_1, \lambda_2])$. Then

$$\Omega(x\gamma y) \geq [\lambda_1, \lambda_2] = \Omega(x) = \Omega(x) \wedge \Omega(y).$$

Case(II): If $\Omega(x) \geq \Omega(y)$. Let $[\lambda_1, \lambda_2] = \Omega(y)$. Then $y \in (\Omega, [\lambda_1, \lambda_2])$.
By assumption we've $(\Omega, [\lambda_1, \lambda_2])$ is an gI of S. So $x\gamma y \in (\Omega, [\lambda_1, \lambda_2])$

$$\Omega(x\gamma y) \geq [\lambda_1, \lambda_2] = \Omega(y) = \Omega(x) \wedge \Omega(y).$$

Hence from Case(I) and Case(II) we have $\Omega(x\gamma y) \geq \Omega(x) \wedge \Omega(y)$. Therefore Ω is an $IFgI$ of S. Similarly we prove for other case. □

Theorem 3.5. *Let Ω be an $IFgI$ of S. Then $\partial^u(\Omega)$ is an $IFgI$ of S.*

Proof. Assume that Ω is an $IFgI$ of S. Let $x, y \in S$ and $\gamma \in \Gamma$ then
$\Omega(x\gamma y) \geq \Omega(x) \vee \Omega(y)$.

$$\begin{aligned}
\partial^u(\Omega)(x\gamma y) &= \bigvee_{a \in [x\gamma y]_\partial} \Omega(a) \\
&= \bigvee_{a \in [x]_\partial \gamma [y]_\partial} \Omega(a) \\
&= \bigvee_{c \in [x]_\partial d \in [y]_\partial} \Omega(c\gamma d) \\
&\geq \left(\bigvee_{c \in [x]_\partial} \Omega(c) \right) \wedge \left(\bigvee_{d \in [y]_\partial} \Omega(d) \right) \\
&= \partial^u(\Omega)(x) \wedge \partial^u(\Omega)(y).
\end{aligned}$$

Hence the theorem. □

An IF subset of S is called an IRF sub-Γ-semigroup of S if it is both upper and lower IRF sub-Γ-semigroup of S.

Theorem 3.6. *If Ω is an IF sub-Γ-semigroup of S, then $\partial^u(\Omega)$ is an IF sub-Γ-semigroup of S.*

Proof. Assume that Ω is an IF sub-Γ-semigroup of S. Let $x, y \in S$ and $\gamma \in \Gamma$. Then $\Omega(x\gamma y) \geq \Omega(x) \wedge \Omega(y)$

$$\begin{aligned}
\partial^u(\Omega)(x\gamma y) &= \bigvee_{p\gamma q \in [x\gamma y]_\partial} \Omega(p\gamma q) \\
&\geq \bigvee_{p \in [x]_\partial \gamma q \in [y]_\partial} (\Omega(p\gamma q) \\
&\geq \bigvee_{p \in [x]_\partial \gamma q \in [y]_\partial} (\Omega(p) \wedge \Omega(q)) \\
&\geq (\bigvee_{p \in [x]_\partial} \Omega(p)) \wedge (\bigvee_{q \in [x]_\partial} \Omega(q)) \\
&= \partial^u(\Omega)(x) \wedge \partial^u(\Omega)(y)
\end{aligned}$$

This proves that $\partial^u(\Omega)$ is an IF sub-Γ-semigroup of S. Similarly we can prove for lower approximation. □

Theorem 3.7. *If Ω is an IF sub-Γ-semigroup of S, then $\partial^l(\Omega)$ is an IF sub-Γ-semigroup of S.*

By combining Theorem 3.6 and Theorem 3.7 we get the following theorem.

Theorem 3.8. *Let Ω be an IF sub-Γ-semigroup of S. Then Ω is an IRF sub-Γ-semigroup of S.*

Theorem 3.9. *Let Ω be an IF subset of S then Ω is an IF sub-Γ-semigroup of S if and only if $(\Omega, (\lambda_1, \lambda_2))$ is a sub-Γ-semigroup of S for all $[\lambda_1, \lambda_2] \in D[0, 1]$.*

Proof. Suppose Ω is an IF sub-Γ-semigroup of S. Let $x, y \in (\Omega, (\lambda_1, \lambda_2))$ and $\gamma \in \Gamma$ then

$$\Omega(x) > (\lambda_1, \lambda_2) \text{ and } \Omega(y) > (\lambda_1, \lambda_2) \text{ implies } \Omega(x\gamma y) > (\lambda_1, \lambda_2)$$

Thus $x\gamma y \in (\Omega, (\lambda_1, \lambda_2))$. Hence $(\Omega, (\lambda_1, \lambda_2))$ is a sub-Γ-semigroup of S.
Conversely, suppose that $(\Omega, (\lambda_1, \lambda_2))$ is a sub-Γ-semigroup of S for each $[\lambda_1, \lambda_2] \in D[0, 1]$. For any $x, y \in S$, and $\gamma \in \Gamma$. Let

$$\Omega(x) = [\lambda_1, \lambda_2] \text{ and } \Omega(y) = [\lambda_3, \lambda_4].$$

Then $\Omega(x) = (\lambda_1, \lambda_2) > (\lambda_1 \wedge \lambda_3, \lambda_2 \wedge \lambda_4)$ and $\Omega(y) = (\lambda_3, \lambda_4) > (\lambda_1 \wedge \lambda_3, \lambda_2 \wedge \lambda_4)$. Thus $x, y \in (\Omega, (\lambda_1 \wedge \lambda_3, \lambda_2 \wedge \lambda_4))$, since $(\lambda_1 \wedge \lambda_3, \lambda_2 \wedge \lambda_4) \in D[0, 1]$. By hypothesis $x\gamma y \in (\Omega, (\lambda_1 \wedge \lambda_3, \lambda_2 \wedge \lambda_4))$. Hence Ω is an IF sub-Γ-semigroup of S. □

Theorem 3.10. *Let Ω be an IF subset of S then Ω is an IF sub-Γ-semigroup of S if and only if $(\Omega, [\lambda_1, \lambda_2])$ is a sub-Γ-semigroup of S for all $[\lambda_1, \lambda_2] \in D[0, 1]$.*

Theorem 3.11. *If Ω is an $IFBgI$ of S, then $\partial^u(\Omega)$ is an $IFBgI$ of S.*

Proof. Assume that Ω is an $IFBgI$ of S. Then $\Omega(x\beta s\gamma y) \geq \Omega(x) \wedge \Omega(y)$ for all $x, y, s \in S$ and $\beta, \gamma \in \Gamma$.

$$\begin{aligned}
\partial^u(\Omega)(x\beta s\gamma y) &= \bigvee_{r \in [x\beta s\gamma y]_\partial} \Omega(r) \\
&= \bigvee_{r \in [x]_\partial \beta [s]_\partial \gamma [y]_\partial} \Omega(r)
\end{aligned}$$

$$
\begin{aligned}
&= \bigvee_{pqt\in[x]_\partial\beta[s]_\partial\gamma[y]_\partial} \Omega(p\beta q\gamma t) \\
&\geq \bigvee_{p\in[x]_\partial t\in[y]_\partial} [(\Omega(p)\wedge\Omega(t))] \\
&= \partial^u(\Omega)(x)\wedge\partial^u(\Omega)(y) \qquad (1)
\end{aligned}
$$

From equation (1) and by Theorem 3.1 we obtain $\partial^u(\Omega)$ is an $IFBgI$ of S. □

Theorem 3.12. *If Ω is an $IFBgI$ of S, then $\partial^l(\Omega)$ is an $IFBgI$ of S.*

Theorem 3.13. *Let Ω be an IF subset of S. If Ω is an $IFBgI$ of S, then Ω is an $IRFBgI$ of S.*

Proof. Follows from Theorem 3.11 and 3.12 □

Theorem 3.14. *An IF subset of S is an $IFBgI$ of S if and only if $(\Omega,(\lambda_1,\lambda_2))$ is a BgI of S for all $[\lambda_1,\lambda_2]\in D[0,1]$.*

Proof. Suppose that Ω is an $IFBgI$ of S. Let $[\lambda_1,\lambda_2] \in D[0,1]$. Then Ω is an IF sub-Γ-semigroup of S. By Theorem 3.9 $(\Omega,(\lambda_1,\lambda_2))$ is a sub-Γ-semigroup of S. Let $t \in (\Omega,(\lambda_1,\lambda_2))\Gamma S\Gamma(\Omega,(\lambda_1,\lambda_2))$,$y \in S$ and $\alpha,\beta\in\Gamma$ such that $t = x\alpha y\beta z$. Since Ω is an $IFBgI$, $\Omega(x\alpha y\beta z)\geq\Omega(x)\wedge\Omega(z) > (\lambda_1,\lambda_2)$ implies $t\in(\Omega,(\lambda_1,\lambda_2))$.
Thus $(\Omega,(\lambda_1,\lambda_2))\Gamma S\Gamma(\Omega,(\lambda_1,\lambda_2))\subseteq(\Omega,(\lambda_1,\lambda_2))$. Hence $(\Omega,(\lambda_1,\lambda_2))$ is a BgI of S.
Conversely, suppose that $(\Omega,(\lambda_1,\lambda_2))$ is a BgI of S, then $(\Omega,(\lambda_1,\lambda_2))$ is sub-Γ-semigroup of S. By Theorem 3.9 Ω is an IF sub-Γ-semigroup of S. For any $x,z\in S$ and $\alpha,\beta\in\Gamma$.
Let $\Omega(x)=[\lambda_3,\lambda_4]$ and $\Omega(z)=[\lambda_5,\lambda_6]$ then $x,z\in(\Omega,(\lambda_3\wedge\lambda_5,\lambda_4\wedge\lambda_6))$.
Let $y\in S$ then $x\alpha y\beta z\in(\Omega,(\lambda_3\wedge\lambda_5,\lambda_4\wedge\lambda_6))$ implies that
$\Omega(x\alpha y\beta z) > (\lambda_3\wedge\lambda_5,\lambda_4\wedge\lambda_6)=\Omega(x)\wedge\Omega(z)$. Hence Ω is an $IFBgI$ of S. □

4. Interval Rough fuzzy Prime-Γ-ideal($IRFPgI$)

In this section we introduce the concept of $IRFPgI$ of S.

Theorem 4.1. *Let Ω be an IF subset of S. Then Ω is an $IFPgI$ of S if and only if for all $[\lambda_1,\lambda_2]\in D[0,1]$, $(\Omega,[\lambda_1,\lambda_2])$ is a PgI of S.*

Proof. Assume that Ω is an $IFPgI$ of S. Then Ω is an IF an gI of S. Assume $(\Omega,[\lambda_1,\lambda_2])\neq\phi$. By Theorem 3.4 $(\Omega,[\lambda_1,\lambda_2])$ is a gI of S. Let $x,y\in S$ and $\gamma\in\Gamma$ then $x\gamma y\in\Omega_{[\lambda_1,\lambda_2]}$. Since Ω is an $IFPgI$ of S. $\Omega(x\gamma y)=\Omega(x)$ or $\Omega(x\gamma y)=\Omega(y)$ and $x\in(\Omega,[\lambda_1,\lambda_2])$ or $y\in(\Omega,[\lambda_1,\lambda_2])$ Therefore $(\Omega,[\lambda_1,\lambda_2])$ is a PgI of S.

Conversely, assume that for all $[\lambda_1,\lambda_2]\in D[0,1]$, if $(\Omega,[\lambda_1,\lambda_2])\neq\phi$, then $(\Omega,[\lambda_1,\lambda_2])$ is a PgI of S. Let $x,y\in S$ and $\gamma\in\Gamma$. By Theorem 3.4 Ω is an $IFgI$ of S. This implies $\Omega(x\gamma y)\geq\Omega(y)$ and $\Omega(x\gamma y)\geq\Omega(x)$. Let $\lambda_1=\Omega(x\gamma y)$. Thus $x\gamma y\in(\Omega,[\lambda_1,\lambda_2])$. Since $(\Omega,[\lambda_1,\lambda_2])$ is a PgI of S, $x\in(\Omega,[\lambda_1,\lambda_2])$ or $y\in(\Omega,[\lambda_1,\lambda_2])$. This implies that $\Omega(x)\geq[\lambda_1,\lambda_2]=\Omega(x\gamma y)$ or $\Omega(y)\geq[\lambda_1,\lambda_2]=\Omega(x\gamma y)$. Hence $\Omega(x\gamma y)=\Omega(x)$ or $\Omega(x\gamma y)=\Omega(y)$. Hence Ω is an $IFPgI$ of S. □

Theorem 4.2. *Let Ω be an IF subset of S. Then Ω is an $IFPgI$ of S if and only if for all $[\lambda_1,\lambda_2]\in D[0,1]$, $(\Omega,(\lambda_1,\lambda_2))$ is a PgI of S.*

Theorem 4.3. *Let Ω be an $IFPgI$ of S. Then Ω is an $IRFPgI$ of S.*

Proof. By applying Theorems 3.4, 4.2, 2.1 and 2.2 we get the result. □

5. INTERVAL ROUGH FUZZY PRIME-BI-Γ-IDEAL ($IRFPBgI$)

In this section we introduce the notion of $IRFPBgI$ of S also discuss some property of this ideal.

Theorem 5.1. *Let Ω be an IF subset of S. Then Ω is an $IFPBgI$ of S if and only if $(\Omega, [\lambda_1, \lambda_2])(resp., (\Omega, (\lambda_1, \lambda_2)) \neq \phi)$ is a $PBgI$ of S for every $[\lambda_1, \lambda_2] \in D[0, 1]$.*

Proof. Suppose that Ω is an $IFPBgI$ of S. Then Ω is an $IFBgI$ of S. Assume that $(\Omega, [\lambda_1, \lambda_2]) \neq \phi$. By Theorem 3.14 $(\Omega, [\lambda_1, \lambda_2])$ is a BgI of S. Let $x, y, a \in S$ and $\gamma, \beta \in \Gamma$ such that $x\gamma a\beta y \in (\Omega, [\lambda_1, \lambda_2])$. Since Ω ia an $IFPBgI$ of S, we have $\Omega(x\gamma a\beta y) = \Omega(x)$ or $\Omega(x\gamma a\beta y) = \Omega(x)$. Thus $x \in (\Omega, [\lambda_1, \lambda_2])$ or $y \in (\Omega, [\lambda_1, \lambda_2])$. Hence $(\Omega, [\lambda_1, \lambda_2])$ is a $PBgI$ of S.

Conversely, suppose that for all $[\lambda_1, \lambda_2] \in D[0, 1]$, if $(\Omega, [\lambda_1, \lambda_2]) \neq \phi$, and $(\Omega, [\lambda_1, \lambda_2])$ is a $PBgI$ of S. Let $x, a, y \in S$ and $\gamma, \beta \in \Gamma$. By Theorem 3.14 Ω is an $IFPBgI$ of S. Then, we have $\Omega(x\gamma a\beta y) = \Omega(x) \wedge \Omega(y)$.
Let $[\lambda_1, \lambda_2]$ = $\Omega(x\gamma a\beta y)$ and $x\gamma a\beta y \in (\Omega, [\lambda_1, \lambda_2])$. Since $(\Omega, [\lambda_1, \lambda_2])$ is a $PBgI$ of S, we have $x \in (\Omega, [\lambda_1, \lambda_2])$ or $y \in (\Omega, [\lambda_1, \lambda_2])$ which implies

$\Omega(x) \geq [\lambda_1, \lambda_2] = \Omega(x\gamma a\beta y)$ or $\Omega(x) \geq [\lambda_1, \lambda_2] = \Omega(x\gamma a\beta y)$.

Hence Ω is an $IFPBgI$ of S. □

A nonempty set IF set Ω is called an $IRFPBgI$ of S if it is both lower and upper $IRFPBgI$ of S.

Theorem 5.2. *Let Ω be an $IFPBgI$ of S. Then Ω is an $IRFPBgI$ of S.*

Theorem 5.3. *Let Ω be an $IFPBgI$ of S. Then Ω is a ∂-lower $IRFPBgI$ of S if and only if for all $[\lambda_1, \lambda_2] \in D[0, 1]$ $\underline{\partial}(\Omega, [\lambda_1, \lambda_2]) \neq \phi$, and $(\Omega, [\lambda_1, \lambda_2])$ is a ∂-lower rough $PBgI$ of S.*

Proof. Suppose that Ω is a ∂-lower $IRFPBgI$ of S. $\underline{\partial}(\Omega)$ is a $IFPBgI$ of S. By Theorem 5.1 $(\underline{\partial}(\Omega), [\lambda_1, \lambda_2])$ is a $PBgI$ of S. By Theorem 2.1 $\underline{\partial}(\Omega, [\lambda_1, \lambda_2])$=$(\underline{\partial}(\Omega), [\lambda_1, \lambda_2])$ and this implies $\underline{\partial}(\Omega, [\lambda_1, \lambda_2])$ is a $PBgI$ of S. Hence $(\Omega, [\lambda_1, \lambda_2])$ is a ∂-lower rough $PBgI$ of S. Similarly, we can obtain the converse part. □

Theorem 5.4. *Let Ω be an $IFPBgI$ of S. Then Ω is a ∂-upper $IRFPBgI$ of S if and only if for all*
$[\lambda_1, \lambda_2] \in D[0, 1]$ $(\Omega, (\lambda_1, \lambda_2)) \neq \phi$, and $(\Omega, (\lambda_1, \lambda_2))$ is a ∂-upper rough $PBgI$ of S.

Proof. The proof is similar to Theorem 5.3 and follows from Theorem 2.1 we can obtain the proof easily. □

6. CONCLUSION

Fuzzy set theory and rough set theory take into account two distinct aspects of uncertainty that can be experienced in real world problems in many fields. The fusion of fuzzy set and rough set lead to various models. This paper is deliberated to built up a relation between rough sets, fuzzy sets and interval-valued fuzzy sets. In the present paper, we use Γ-semigroup instead of universe set, and introduced the notion of interval-valued rough fuzzy ideals, interval-valued fuzzy bi-ideals and interval-valued fuzzy prime ideals. Also, we believe, this paper will turn out to be more useful in the theory of rough sets and fuzzy sets.

7. Acknowledgements

The authors would like to thank to the referees for a number of constructive comments and valuable suggestions.

References

[1] R. Chinram. Rough prime ideals and rough fuzzy prime ideals in gamma semigroups, Commun. Korean. Math. Soc, 24(3),(2009), 341-387.
[2] D. Dubois and H.Prade. Rough fuzzy sets and fuzzy rough sets. Int.J.General Syst, 17(2-3), (1990), 191-209.
[3] Z. Gong, B. Sun and D. Chen. Rough set theory for the interval valued fuzzy information systems, Information science, 178, (2008), 1968-1985.
[4] Z. Pawlak. Rough sets. Int.J.Inform.comput science, 11,(1982), 341-356,
[5] V. S. Subha, N. Thillaigovindan and P. Dhanalakshmi. Interval valued rough fuzzy ideals in semigroups. Journal of Emerging Technologies and Innovative Research, 6(3), (2019), 271-276,
[6] V. S. Subha, V.Chinnadurai and P. Dhanalakshmi. Characterizations of semigroups by rough interval pythagorean fuzzy sets. TWMS Journal of Applied and Engineering Mathematics (communicated).
[7] M. K. Sen. On Γ-semigroups. Algebra and its applications (New Delhi, 1981), Lecture note in Pure and Appl. Math., 91, Dekker, New York, 1984,
[8] N. Thillaigovindan, V. Chinnadurai and V. S. Subha. On rough ideals in Γ-semigroups. International Journal of Algebra, 14(6), (2012), 651-661,
[9] L. A. Zadeh. Fuzzy sets. Information control, 8, (1965), 338-353,
[10] L. A. Zadeh. The concept of a linguistic variable and it's application to approximation reasoning I. Information Sciences, 8, (1975), 199-249,

V. S. SUBHA
Dharmapuram Gnanambigai Govt. Arts college(W), Mayiladuthurai-609001
Email address: dharshinisuresh2002@gmail.com

P. DHANALAKSHMI
Annamalai University, Annamalai Nagar-608002
Email address: vpdhanam83@gmail.com